A. **First–Order Linear Differential Equations.**

A differential equation is **first–order linear** if it can be put into form:

and is further **homogeneous** if:

Remember first–order means only up to the first derivative appears. Linear means y and y' appear linearly.

We call this form **standard form**.

For example:

$y' - 2y = e^{3x}$

This example is not homogeneous, aka is **inhomogenous**.

One strategy for solving is to **multiply** by an oh-so-special **integrating factor** $I(x)$ that converts the **lefthand side** to form:

An integrating factor for $y' - 2y$ turns out to be $I(x) = e^{-2x}$. Let's use it to solve:

$y' - 2y = e^{3x}$

Don't worry my friend. Later we will see how to find the integrating factor.

After we multiply by the integrating factor the remaining steps are:

1. Integrate and don't forget the $+C$!
2. Solve for y.

The general format of solutions to a 1st–order linear differential equation is:

$y = y_p + Cy_h$

where y_p is a **particular solution** and y_h is a **homogeneous solution**.

The homogeneous solution is so–called because it solves the associated homogeneous equation $y' + a(x)y = 0$.

B. **Finding the Integrating Factor.** Let us find the integrating factor for:

$y' + a(x)y$

Remember the goal is that multiplying the expression by $I(x)$ should yield $I(x)y' + I'(x)y$.

The integrating factor for $y' + a(x)y$ is:

$I(x) =$

Example 1. Solve the initial value problem:

$ty' + 3y = \frac{1}{te^t}$ with $y(-1) = 5$

To use our formula to find an integrating factor first we need to convert to standard form.

C. **Variation of Parameters.** Let us reassess:

$y' + a(t)y = f(t)$

A **fundamental homogeneous** solution is:

$y_h =$

We will next identity the general solution by writing it in form:

$y =$

Let us solve for $u(t)$ by plugging into: $y' + a(t)y = f(t)$.

Much of math is about viewing the same problem from different perspectives for the purpose of obtaining new insights!

A **fundamental homogeneous solution** is a nonzero solution y_h to the associated homogeneous equation $y' + a(t)y = 0$. The formula to the left is derived in discussion section.

We call $u(t)$ the **variable parameter**. The reason we would expect y to have this form is subtle, and will be revealed much later in the course, when we decompose "force" $f(t)$ into instant "impulses" and think of y_p as the continuous sum (integral) of "responses" to those impulses. Each response happens to be a sudden occurrence of a multiple of y_h.

Variation of Parameters. The general solution to $y' + a(t)y = f(t)$ is:

$y = uy_h$

where:

$y_h = e^{-\int a(t)\,dt}$

$u =$

Example 2. Use variation of parameters to find the general solution to:

$\cos(x)y' = \cos^2(x) - y$ on interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

Remember that:

$\sec x = \dfrac{1}{\cos x}$

and that:

$\int \sec x \, dx = \ln|\sec x + \tan x|$

Note that if $-\frac{\pi}{2} < x < \frac{\pi}{2}$ then:

$\sec x + \tan x = \dfrac{1 + \sin x}{\cos x}$

will be positive as the numerator and denominator will both be positive.