A. Linear Second–Order Differential Equations.

A differential equation is **second–order linear** if it can be put into form:

and is further **homogeneous** if:

Remember second–order means only up to the second derivative appears. Linear means y, y', y'' appear linearly.

We call this form **standard form**. The right side $f(t)$ is commonly referred to as the **forcing term**.

For the time being we will concentrate on the **homogeneous** subcategory.

For example:

$t^2y'' + 3ty' - 3y = 0$

which we verify has among its solutions:

$y_1 = t$

$y_2 = \frac{1}{t^3}$

Functions y_1 and y_2 are **linearly independent** if:

Otherwise they are **linearly dependent**.

Linear independence and linear cominbations are notion from linear algebra. There is a definition of linear independence for a list of more than two functions, but it is a little more complicated. It requires that no function in that list is a linear combination of other functions in that list.

A **linear combination** of y_1 and y_2 has form:

$y =$

Show that if y_1 and y_2 solve the **homogeneous** linear:

$y'' + p(t)y' + q(t)y = 0$

then so does any linear combination $y = C_1y_1 + C_2y_2$.

Be careful: this result only applies in the **homogeneous** case!

A pair of solutions y_1 and y_2 to **homogeneous** 2nd–order linear ODE:

$y'' + p(t)y' + q(t)y = 0$

is called **fundamental set of solutions** if they are:

Theorem: In this case the general solution has form:

$y =$

Again: be warned this only applies in the homogeneous case! And really: added to the hypotheses should be that the functions $p(t)$ and $q(t)$ are "nice" in the sense that they are continuous on the interval in which the solution is being considered.

Example 1. Find the solution to the initial value problem:

$t^2y'' + 3ty' - 3y = 0$ with $y(1) = 0$ and $y'(1) = 4$

using that a fundamental set of solutions $y_1 = t$ and $y_2 = \frac{1}{t^3}$.

B. **Wronskian.**

The **Wronskian** of function y_1 and y_2 is:

$W(y_1, y_2) =$

This is an example of a **determinant**:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Suppose y_1 and y_2 are linearly dependent: for example $y_2 = cy_1$.

$W(y_1, cy_1) =$

Functions y_1 and y_2 are linearly independent if and only if:

$W(y_1, y_2) \neq$

Really this result is only true for "nice" functions y_1 and y_2. Thankfully, the functions we obtain from solving homogoneous linear differential equations are always "nice" enough.

Suppose now that y_1 and y_2 are solutions to homogeneous:

$y'' + p(t)y' + q(t)y = 0$

Show that $W = W(y_1, y_2)$ satisfies the differential equation:

$(\ln W)' = -p(t)$

Abel's Theorem. If y_1 and y_2 are solutions to:

$y'' + p(t)y' + q(t)y = 0$

then:

$W(y_1, y_2) =$

A function of the form $Ce^{[\text{stuff}]}$ either **always** has value 0 (if $C = 0$) or **never** has value 0 (if $C \neq 0$). This is because exponentials never have value 0.

Example 2. For the solutions $y_1 = t$ and $y_2 = \frac{1}{t^3}$ of the differential equation:

$t^2y'' + 3ty' - 3y = 0$

compute their Wronskian and confirm Abel's Theorem.