A. **Constant Coefficients.**

We specialize to homogeneous 2nd–order LDEs with **constant** coefficients:

LDE is shorthand for linear differential equation.

Let us plug in an exponential trial solution:

$y = e^{\lambda t} \;\mapsto\; y'' + py' + qy = 0$

Trial means we will try it out.

λ is read "lambda".

The **characteristic equation** of $y'' + py' + qy = 0$ is:

and $y = e^{at}$ is a solution of the differential equation if and only if:

Example 1. Find the general solution to the differential equation:

$y'' - 3y' + 2y = 0$

B. **Repeated Roots.** What if the characteristic equation has a repeated root?

If $\lambda = a$ is that repeated root then the differential equation has form:

That root only provides solution $y_1 = e^{at}$. But we have learned that there must be **two** fundamental solutions.

$(\lambda - a)^2 = \lambda^2 - 2a\lambda + a^2$

In discussion you will show that if y_1 is one solution then the other is:

$y_2 = uy_1$

where:

$$u = \int \frac{e^{-\int p(t)\ dt}}{y_1^2}\ dt$$

This was for the differential equation:

$y'' + p(t)y' + q(t)y = 0$

In our case with $y_1 = e^{at}$ we find:

If $y'' + py' + qy = 0$ has repeated root $\lambda = a$ then fundamental solutions are:

$y_1 =$

$y_2 =$

Example 2. Solve the initial value problem:

$y'' - 2y' + y = 0$ with $y(0) = 1$ and $y'(0) = 0$

C. **Complex Roots.** The equation $\lambda^2 + 1 = 0$ has no real roots. What do we do? As a precursor we discuss the interplay of complex exponentials with trig.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

These are Taylor series expansions. Calculus II content!

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots$$

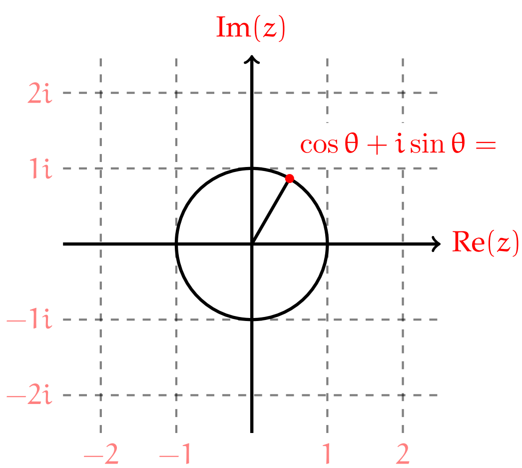
$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

$$e^{i\theta} =$$

Here i is the imaginary number: $i^2 = -1$. So: $i^3 = i^2 i = -i$ and $i^4 = i^2 i^2 = 1$.

Euler's Identity.

$$e^{i\theta} =$$



This is a picture of the complex plane. You would plot the complex number $a + bi$ with coordinates (a, b) corresponding to its real and imaginary parts.

Remember that $(\cos\theta, \sin\theta)$ describes the unit circle.

Suppose that the characteristic equation of $y'' + py' + qy = 0$ has complex roots:

$\lambda = a + bi$ and $\overline{\lambda} = a - bi$

Roots of real polynomials always come in complex conjugate pairs z and $\overline{z}$.

Then its fundamental **complex** solutions are:

Remember (or simply check) that:

$\text{Re}(z) = \frac{1}{2}z + \frac{1}{2}\overline{z}$

$\text{Im}(z) = \frac{1}{2i}z - \frac{1}{2i}\overline{z}$

The take away is that the real and imaginary parts of z are linear combinations of z and $\overline{z}$. And... we know that linear combinations of solutions are also solutions!

Continuing... the complex roots $\lambda = a \pm bi$ yield fundamental **real** solutions:

Example 3. Find the general solution to $y'' - 6y' + 13y = 0$.