A. **Variation of Parameters.** We present another approach to solving:

$(\star)$ $y'' + p(t)y' + q(t)y = f(t)$

using variation of paramaters. Let y_1 and y_2 be solutions to homogeneous:

$(\star_h)$ $y'' + p(t)y' + q(t)y = 0$

We trial solution to $(\star)$ of form:

$y =$

$y' =$

$y'' =$

And next we substitute into $(\star)$ and simplify:

We obtain system:

Which can be solved to yield solution:

Remember 1st–order variation of parameters. It entailed setting $y = uy_h$ where y_h was a homogeneous solution, and then providing a formula for the variable paramater u.

We make the assumption that $u_1'y_1 + u_2'y_2 = 0$ to simplify the process of solving for u_1 and u_2, and to restrict what kind of solutions we get. In reality their are infinitely many combinations of u_1 and u_2 that would work: but we want to hone in one, the one that is simplest to find!

The highlighted terms above vanish when plugged into $(\star)$, because they are linear combinations of homogeneous solutions, hence are homogoneous solutions, meaning plugging them into the lefthand side of $(\star)$ yields 0.

Remember the Wronskian is:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2$$

Variation of Parameters. A particular solution to:

$y'' + p(t)y' + q(t)y = f(t)$

is: $y_p = u_1y_1 + u_2y_2$

where y_1, y_2 are fundamental homogeneous solutions and:

$u_1 =$

$u_2 =$

Applying variation of parameters has two challenges. First: finding the homogeneous solutions y_1 and y_2. Second: computing the requisite integrals. Why does life have so many challenges? Curse you, life!

Example 1. Use variation of parameters to find a particular solution to:

$y'' + y = \tan t$ on interval $-\frac{\pi}{2} < t < \frac{\pi}{2}$

The method of undetermined coefficients will not save us here, because we do not have a trial solution prepared for the forcing term $\tan t$. Although, do you really want undetermined coefficients to be the one who saves you in your time of need?

Remember that:

$\sec x = \dfrac{1}{\cos x}$

and that:

$\int \sec x \; dx = \ln |\sec x + \tan x|$

Note that if $-\frac{\pi}{2} < x < \frac{\pi}{2}$ then:

$\sec x + \tan x = \dfrac{1 + \sin x}{\cos x}$

will be positive as the numerator and denominator will both be positive.

Example 2. Use variation of parameters to find the general solution to:

$y'' - y' - 2y = 3e^{-t}$

I hope that one day I find my variable parameter and that we can have a happy life together.