A. **Laplace Transforms.** The Laplace transform takes a function:

$f(t)$ where t is in the **time–domain**

and transforms it to a function:

$F(s)$ where s is in the **frequency domain**

A differential equation can be effectively solved in the frequency domain.

Afterwards: the solution can be transformed back to the time domain.

The **Laplace transform** of $f(t)$ is:

$F(s) =$

Example 1. Find a formula for:

$\mathcal{L}\{e^{at}\}(s) =$

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The frequency domain consists of complex numbers $s = a + bi$ with positive real part. Intuitively $F(s)$ is designed to explode (get very **large**) at those $s = a + bi$ when $f(t)$ features exponential–periodicity of the form $e^{at} \cos bt$ of $e^{at} \sin bt$. So here: a measures the exponential component, and b measures the frequency.

In terms of limits the Laplace transform is:

$$\lim_{\substack{A \to 0^- \\ B \to \infty}} \int_A^B e^{-st} f(t)\, dt$$

The reason for the 0^- will not be clear until later. With functions that are continuous at 0 there is no need to include the $-$ in 0^-.

The Laplace transform is not always defined, because the improper integral is not always defined. The requirements are $f(t)$ be at least piecewise continuous, and have **exponential order**, meaning there are positive constants M and k so $f(t) \leqslant Me^{kt}$. This guarantees that the Laplace transform is eventually defined, i.e. defined for sufficiently large values of s.

Example 2. Find a formula for:

$$\boxed{\mathcal{L}\{t^n\}(s) =}$$

$\mathcal{L}\{1\}(s) =$

$\mathcal{L}\{t\}(s) =$

$\mathcal{L}\{t^2\}(s) =$

Example 3. Calculate the Laplace transform of:

$$f(t) = \begin{cases} 2 & \text{if } 0 \leqslant t < 2 \\ t-2 & \text{if } t \geqslant 2 \end{cases}$$