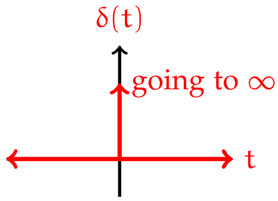
Lecture 18. B4 – Dirac Delta Function and Impulse Response

A. **Dirac Delta Function.**

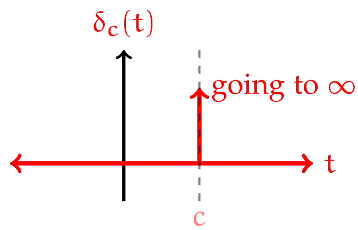
Intuitively the **dirac delta function** is defind by:

$$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases}$$



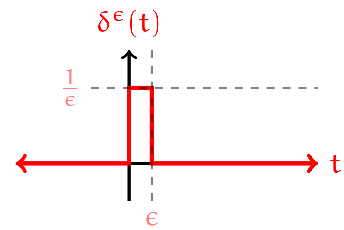
and so its shift right by c is:

$$\delta_c(t) = \delta(t-c) = \begin{cases} & \\ & \end{cases}$$



More precisely $\delta(t)$ is defined to be the limit as $\epsilon \to 0$ of the functions:

$$\delta^\epsilon(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{\epsilon} & \text{if } 0 \leqslant t < \epsilon \\ 0 & \text{if } t \geqslant \epsilon \end{cases}$$



Key Integral Properties. If c is nonnegative:

Total area: $\displaystyle\int_0^\infty \delta(t-c)\ dt =$

Sifting property: $\displaystyle\int_0^\infty \delta(t-c)f(t)\ dt =$

Laplace transform: $\mathcal{L}(\delta_c)(s) =$

$\mathcal{L}(\delta)(s) =$

$\mathcal{L}(\delta_c f)(s) =$

The Laplace transform of δ_c would be:

$$\int_0^\infty e^{-st}\delta_c(t)\ dt$$

Example 1. Solve the following initial value problem:

$y'' - 12y' + 40y = \delta(t - \frac{\pi}{6}) \sin t$ with $y(0) = y'(0) = 0$

Recall the Laplace transform rules:

$\mathcal{L}(y')(s) = sY - y(0)$

$\mathcal{L}(y'')(s) = s^2Y - sy(0) - y'(0)$

Recall the formula from the last page:

$\mathcal{L}(\delta_c f)(s) = e^{-cs} f(c)$

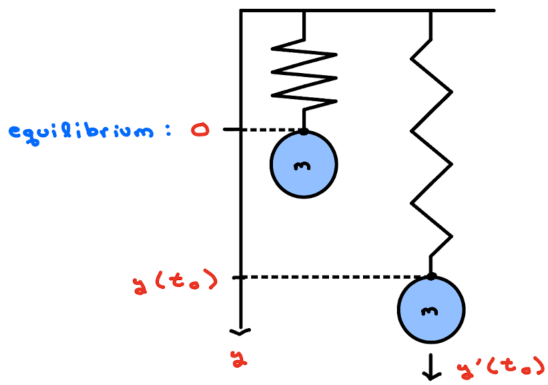
Recall the inverse translation formula:

$\mathcal{L}^{-1}\{e^{-cs}F(s)\}(t) = H_c(t)f(t - c)$

Recall the inverse Laplace formula:

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-a)^2 + b^2}\right\}(t) = \frac{e^{at}\sin bt}{b}$$

B. **Unit Impulse Response.** Consider a mass m hanging from a spring:



In the absence of an external force the differential equation governing motion is:

$my'' + \mu y' + ky = 0$

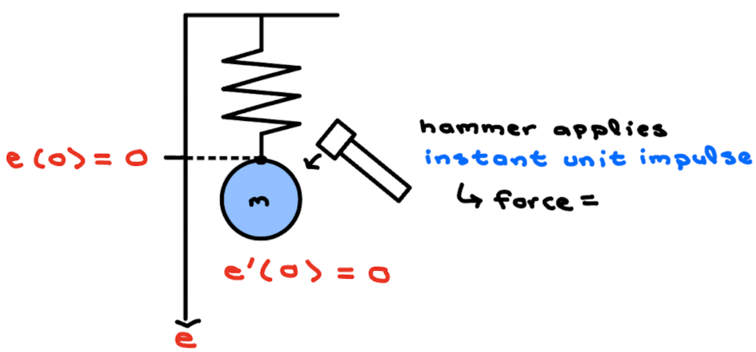
μ is the damping constant and k is the spring constant.

If there **is** an external force $f(t)$ then the appropriate differential equation is:

$my'' + \mu y' + ky =$

In particular consider the response $e(t)$ to an **instant unit impulse**:

Impulse is $(\text{force}) \cdot (\text{change in time})$. If the impulse equals 1, but the change in time is 0, then the force must be ∞.



Given a system with constant coefficients:

$ay'' + by' + cy = f(t)$ with any initial values

the **unit impulse response** is the solution $e(t)$ to initial value problem:

The Laplace transform of this system is:

and its solution is called the **transfer function** E.

$E =$

There is a technicality here: we will see that $e'(0)$ is not actually defined in practice, because the sudden impulse leads to a corner, a failure of differentiability. Therefore the correct condition to impose here is $e'(0^-) = 0$, indicating that there is no response before the impulse hits, i.e. just to the left of time $t = 0$.

In discussion section, you will learn that, in the s–domain, the transfer function converts the forcing term $f(t)$ to a **state–free solution** of the system. A state–free system is one in which the initial values are all 0.

Example 2. Let ω_0 be a constant and consider the system:

$y'' + \omega_0^2 y = \cos t$ with $y(0) = 1$ and $y'(0) = -1$

Find the transfer function and unit impulse response.

Recall the formula:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+b^2}\right\}(t) = \frac{\sin bt}{b}$$

Note that we need to multiply by the Heaviside function $H(t)$ here because there is no response until the impulse hits at $t = 0$. This leads to a strange issue: the condition $e'(0) = 0$ is not technically defined because $e(t)$ will not be differentiable at 0: it will have a corner due to truncation by the Heaviside. So really, the correct initial value to impose here is that $e'(0^-) = 0$, i.e. the derivative coming from the left is zero, as was indicated in an earlier margin note. This is what the impulse reponse will genuinely satisfy.