**Example 1.** Use convolutions and partial fractions to compute the integral:

$$\int_0^t ue^u(t-u)^2 \, du$$

Recall that:

$$f(t) * g(t) = \int_0^t f(u)g(t-u) \, du$$

The idea is to realize that this integral is a convolution, then to convert to multiplication in the $s$–domain using the Laplace transform, and lastly to convert back to the $t$–domain using the inverse Laplace transform.

Recall that:

$$\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

Also recall that:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-a)^n}\right\} = \frac{t^{n-1}e^{at}}{(n-1)!}$$

## A. **Properties of Convolutions.**

**Basic Properties.**

Commutativity: $f * g =$

Linearity: $f * (ag + bh) =$

Vanishing: $f * 0 =$

The first follows from that multiplication in the the $s$–domain is commutative (meaning you can change order freely) and therefore the corresponding operation in the $t$–domain (convolution) is also commutative.

The last two properties follows from linearity properties of integrals.

It is **not** the case that $f * 1 = f$.

A simple check can verify this.

Compute: $(\delta * f)(t) =$

Remember $\delta$ is the dirac delta function. Importantly the sifting property says:

$$\int_0^\infty f(t)\delta(t)\, dt = f(0)$$

**Dirac Delta as Identity.**

$f * \delta = \delta * f =$

Next we consider a convolution product rule for differentation.

$$\frac{d}{dt}\Big[f(t) * g(t)\Big] =$$

We use the multivariable chain rule (calculus III, really?!) for:

$$z = \int_0^x f(u)g(y-u)\, du$$

and $x = y = t$. This yields:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

which because $x = y = t$ simplifies to:

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}$$

The fund. theorem of calculus says:

$$\frac{\partial z}{\partial x} = f(u)g(y-x) = f(t)g(t-t) = f(t)g(0)$$

then derivative under the integral sign:

$$\frac{\partial z}{\partial y} = \int_0^x f(u)g'(y-u)\, du = (f * g')(t)$$

**Convolution Product Rule.**

$$\frac{d}{dt}\Big[f(t) * g(t)\Big] =$$

In particular, if $g(0) = 0$, then $(f * g)' =$

B. **State–Free Response.**

The **state–free response** $y_s$ to a system with constant coefficients:

$ay'' + by' + cy = f(t)$ with any initial values

is the solution $y_s$ with initial values set to $0$.

More generally: **state–free** means the initial values are $0$.

Let $e$ be the impulse response.

Show that $y_s = f * e$ solves the state–free system:

$ay'' + by' + cy = f(t)$ with $y(0) = y'(0) = 0$

Remember, the impulse response is the solution to the initial value problem:

$ae'' + be' + ce = \delta(t)$, $e(0) = e'(0) = 0$

$y_s =$

$y_s' =$

$y_s'' =$

**Formula.** The state–free response for a system is:

$y_s =$

where $e$ is the unit impulse response.

**Example 2.** Let $\omega_0$ be constant, and find an integral formula for the solution to:

$y'' + \omega_0^2 y = g(t)$ with $y(0) = y'(0) = 0$

In the last example from B4 we found the impulse response for this system to be:

$e = \dfrac{1}{\omega_0}\sin(\omega_0 t)$

Refer to that example for the details.