A. **Input Free Response.**

The **input–free response** y_i to a system:

$ay'' + by' + cy = f(t)$ with initial values $y(0) = y_0$ and $y'(0) = y_1$

is the solution y_i with forcing term $f(t)$ set to 0.

Show that the solution to the initial value problem above is:

$y = y_s + y_i$

$y = y_s + y_i$ is the sum of the state–free (initial values set to 0) and input–free (forcing term set to 0) responses.

Recall: the state–free solution y_s satisfies:

$ay_s'' + by_s + cy_s = f(t)$, $y_s(0) = y_s'(0) = 0$

Let us find a formula for the input–free response.

$ay_i'' + by_i' + cy_i = 0$ with initial values $y_i(0) = y_0$ and $y_i'(0) = y_1$

The transfer function in this case will be:

$E = \dfrac{1}{as^2 + bs + c}$

Applying the Laplace transform to the input–free system and simplifying yields:

$Y_i = AsE + BE$ for constants A and B

We want to apply the Laplace inverse. We do this by noting that:

$\mathcal{L}(e') = sE - se(0) = sE$

which tells us $\mathcal{L}^{-1}\{sE\} = e'$.

If you are observant you may wonder how $y_i(t)$ can equal $Ae'(t) + Be(t)$ since isn't $e(0) = e'(0) = 0$ as part of being the impulse response? But that would mean $y_i(0) = 0\ldots$ Well here is where we recall that in practice $e'(0)$ is actually undefined (as a sudden impulse leads to a corner) and it was really $e'(0^-)$ that equalled 0. It is the values–from–the–right $y_i(0^+)$ and $y_i'(0^+)$ that will equal y_0 and y_1.

General Solution Formula. The solution to:

$ay'' + by' + cy = f(t)$ with initial values $y(0) = y_0$ and $y'(0) = y_1$

is $y = y_s + y_i = e * f +$

where e is the unit impulse response.

Example 1. Provide an integral formula for the solution to:

$y'' + 4y' + 13y = g(t)$ with $y(0) = -6$ and $y'(0) = 3$

Recall:

$$E = \frac{1}{as^2 + bs + c}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-a)^2 + b^2}\right\} = \frac{e^{at}\sin bt}{b}$$

Recall the input-free response to:

$ay'' + by' + cy = [\text{anything}]$

with initial values: $y(0) = y_0$, $y'(0) = y_1$

is: $y_i = ay_0e'(t) + (ay_1 + by_0)e(t)$