A. **Linear Systems.** We illustrate the process of solving a linear system:

$$\begin{cases} x + 2y + 3z = 0 \\ -3x - 5y - 8z = 8 \\ 2x + 6y + z = 2 \end{cases}$$

A **linear system** is a system of equations in which variables appear linearly. In other words: each side of the equation is a linear combination of variables and constants. If you forgot: a linear combination of variables x, y, z for example has form $ax + by + cz$ where a, b, and c are constant.

First we convert to an **augmented matrix** with variable coefficients on the left and constants on the right.

A **matrix** is a rectangular array of numbers.

Next we apply **Gaussian elimination** to solve.

It involves the following **row operations**:

swap:

scale:

linearly combine:

The first nonzero entry in each row is called a **pivot**.

Step 1. Use row operations to make all entries below the top pivot equal 0.

This has the effect of **eliminating** x from the bottom equations.

Step 2. Use row operations to make all entries below the next highest pivot equal 0, without upsetting the first column.

Step 3. Finish solving using **back–substitution**.

In other words, solve the bottom equation, then substitute the solution into the next–from–bottom equation, and so on.

B. **Free Variables.** In the previous example if we had made the adjustment:

$$\begin{cases} x+2y+3z=0 \\ -3x-5y-8z=8 \\ 2x+6y+\boxed{z}=\boxed{2} \end{cases} \mapsto \begin{cases} x+2y+3z=0 \\ -3x-5y-8z=8 \\ 2x+6y+\boxed{8z}=\boxed{17} \end{cases}$$

The end of Gaussian elimination would have lead to:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 8 \\ 0 & 0 & & \end{array}\right)$$

which is **inconsistent** because it has:

The idea is that the equation $0=1$ is unsatisfiable.

On the other hand if we had instead made the adjustment:

$$\begin{cases} x+2y+3z=0 \\ -3x-5y-8z=8 \\ 2x+6y+\boxed{z}=\boxed{2} \end{cases} \mapsto \begin{cases} x+2y+3z=0 \\ -3x-5y-8z=8 \\ 2x+6y+\boxed{8z}=\boxed{16} \end{cases}$$

The end of Gaussian elimination would have lead to:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 8 \\ 0 & 0 & & \end{array}\right)$$

The system is **consistent** but:

It is consistent because there are no unsatisfiable equations at the end of our Gaussian elimination.

In this case: assign every variable whose column has no pivot as a **free variable**, and use back–substitution to solve for the others in terms of free variables.

A free variable can take on any value, meaning there will be ∞ solutions to the system.

The goal of Gaussian elimination is to use row–operations to convert an augmented matrix to **row–echelon form**. This form is characterized by:

1. Each pivot is:

2. All rows of all zeroes are:

C. **Matrix Operations.** In general a **matrix** is a rectangular array of numbers.

Of special note are (column) **vectors** which contain only one column:

For us, a vector will always refer to a column vector. But it in many contexts it is perfectly reasonable to talk about row vectors too.

There are number of linear operations we can execute with matrices.

scalar multiplication: $5\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} =$

A real number is referred to as a **scalar**. In scalar multiplication, each entry is multiplied (scaled) by that scalar.

addition: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} =$

Only matrices of the same **size** (same number of rows, same number of columns) can be added. We simply add the entries located in the same positions.

transpose: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^T =$

Taking the transpose swaps rows and columns.

zero matrix: $0 =$

Properties. Let A and B be matrices of the same size. Let k be a scalar.

commutativity of addition: $A + B =$

distributivity of scalar multiplication: $k(A + B) =$

zero matrix as additive identity: $A + 0 =$

D. **Matrix Product.** We define (matrix)(column vector) so we can convert:

$$\begin{cases} x + 2y + 3z = 0 \\ -3x - 5y - 8z = 8 \\ 2x + 6y + z = 2 \end{cases} \mapsto$$

The expression on the right is referred to as the **matrix form** of the system: $A\mathbf{x} = \mathbf{b}$.

In other words:

$$\begin{pmatrix} \text{row } 1 \\ \vdots \\ \text{row } m \end{pmatrix} (\text{column vector}) =$$

Here $\cdot$ is the **dot product** which you learned in Calculus III. If you forgot:

$\langle a_1, \ldots, a_n \rangle \cdot \langle b_1, \ldots, b_n \rangle$

$= a_1 b_1 + \cdots + a_n b_n$

And then we define product (matrix)(matrix) by:

$$\begin{pmatrix} \text{row } 1 \\ \vdots \\ \text{row } m \end{pmatrix} \begin{pmatrix} \text{column } 1 & \cdots & \text{column } n \end{pmatrix} =$$

The matrix product is not always defined. It requires that we take the dot product of rows of the first matrix with columns of the second. But a dot product can only be taken of vectors with the same number of entries.

or equivalently:

(entry in row i and column j of product) $=$

For example:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} =$$

The **identity matrix** I has 1s along the main diagonal and 0s elsewhere:

$I =$

and has the property that:

$IA =$

$AI =$

The **main diagonal** is the diagonal from the upper–left to the bottom–right.

Here A is a matrix so that the multiplication is defined.

Example 1. Consider the matrices.

$D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Compute the following, which model pecularities of the matrix product.

$DN =$

$ND =$

The matrix product is **not** generally commutative. That is, it is possible that:

AB

$N^2 =$

The matrix product does **not** have the cancellation property. That is:

$AB = 0$ and $A \neq 0$ does not imply:

A matrix N is **nilpotent** if there is a positive integer k so:

Certainly there is no nonzero real number that, when raised to a power, yields 0. Matrices are weird.

Nonetheless, the matrix product does have some reasonable properties.

Matrix Product Properties. Let A, B, C be matrices and k be a scalar.

commuativity with scalar multiplication: $k(AB) =$

distributivity: $A(B + C) =$

$(A + B)C =$

associativity: $A(BC) =$

multiplication with zero matrix: $A0 =$