A. **Matrix Inverses.** We said that the matrix product does not have the cancellation property. However there are **special** matrices that can be cancelled.

A matrix $A$ is **invertible** if it has an **inverse** matrix $A^{-1}$ so that:

The inverse is like the 'cancelling' of $A$.

**Inverses of $2 \times 2$ Matrices.**

The inverse of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $A^{-1} =$

The denominator is called the **determinant** of $A$:

$\det A = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} =$

and the matrix is invertible if and only if:

$2 \times 2$ means $2$ rows and $2$ columns.

Note in the formula for the inverse: the entries on the main diagonal (from upper–left to bottom–right) are swapped, and the other entries have their sign flipped.

**Example 1.** Use matrix inverses to solve the system:

$$\begin{cases} 2x + 3y = 5 \\ 3x + 9y = 2 \end{cases}$$

The general strategy is:

**Step 1.** Write in matrix form $A\mathbf{x} = \mathbf{b}$.

**Step 2.** Find $A^{-1}$ if possible. If not possible, since $\det A = 0$, then the system either is inconsistent or has free variables.

**Step 3.** Multiply the equation by $A^{-1}$ to solve for the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

## B. **Eigenstuff.**

An **eigenvector** of a square matrix $A$ is a **nonzero** column vector $\mathbf{v}$ to that:

$A\mathbf{v} =$

where $\lambda$ is a scalar called the **eigenvalue** of $A$ corresponding to $\mathbf{v}$.

**Square** means the same number of rows as columns.

The definition may seem random now, but will turn out to be very useful in solving systems of differential equations.

In words: multiplying $A$ by an eigenvector scales the eigenvalue by a factor of $\lambda$.

For example confirm that: $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector of $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$

How do we locate eigenvalues and eigenvectors? We solve:

$A\mathbf{x} = \lambda\mathbf{x}$

The idea is that $(A - \lambda I)\mathbf{x} = \mathbf{0}$ always has solution $\mathbf{x} = \mathbf{0}$. For it to have another solution (which would be an **eigenvector** of $A$) then $A - \lambda I$ cannot be invertible, otherwise we could 'cancel' it to conclude it must be the case that $\mathbf{x} = \mathbf{0}$.

**Finding Eigenstuff.**

The **eiegenvalues** are $A$ are solutions to the **characteristic equation**:

and the corresponding eigenvectors are found by solving:

We call $\det(A - \lambda I)$ the **characteristic polynomial** because, as we will see, it turns out to be a polynomial in variable $\lambda$.

**Example 2.** Find the eigenvalues of $A$ and an eigenvector for each eigenvalue:

$$A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$$

**Step 1.** Solve the characteristic equation $\det(A - \lambda I) = 0$ to find the eigenvalues.

**Step 2.** For each eigenvalue $\lambda = c$, find a nonzero solution to $(A - cI)\mathbf{x} = \mathbf{0}$. There should always be a free variable, and you can find an eigenvector by setting the free variable to any **nonzero** value you like. There are infinite choices of eigenvectors, obtained by selecting different values for the free variable.