A. **Generalized Eigenvectors.** Suppose $2 \times 2$ matrix $A$ has does **not** have an eigenbasis, only having one linearly independent eigenvector $\mathbf{v}$, and let $\lambda = c$ be the associated eigenvalue.

Due to theoretical principles the following equation will be consistent:

Remember: to be consistent, means has a solution.

In the above scenario, a vector $\mathbf{v}_g$ is a **generalized eigenvector** associated to eigenvector $\mathbf{v}$ if:

The point of having a generalized eigenvector will be to solve systems of differential equations. One linearly independent eigenvector is not enough to fully describe the solution, and it turns out that what we need would then be an associated generalized eigenvector.

**Example 1.** Find a generalized eigenbasis for:

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$$

A **generalized eigenbasis** of a $2 \times 2$ matrix consists of an eigenvector and an associated generalized eigenvector.

Earlier you found this $A$ has: eigenvalue $\lambda = 2$ with eigenvector $\mathbf{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

B. **Systems of 1st–Order Differential Equations.**

Verify that $x(t) = 2e^{-t}$ and $y(t) = e^{-t}$ solves the initial value problem:

$$\begin{cases} x' = -4x + 6y \\ y' = -3x + 5y \end{cases} \quad \text{with:} \quad \begin{cases} x(0) = 2 \\ y(0) = 1 \end{cases}$$

First–order initial value problems can come in various forms.

Unless otherwise stated, the independent variable is $t$.

$$\begin{cases} x' = x \cos z \\ y' = x^2 e^y \\ z' = x \sin y \end{cases} \quad \text{with:} \quad \begin{cases} x(0) = 1 \\ y(0) = -1 \\ z(0) = 0 \end{cases}$$

or:

$$\begin{cases} x' = tx \\ y' = -3x - 2y + 5\cos t \end{cases} \quad \text{with:} \quad \begin{cases} x(1) = 0 \\ y(1) = 1 \end{cases}$$

A 1st–order initial value problem can be expressed in **vector form**:

and is:

- **autonomous** if:
- **linear** if:
- **autonomous** if:

For the dependent variables to appear linearly, each entry of $\mathbf{f}(\mathbf{x}, t)$ should look be a sum of terms of the form:

[constant or function of t] · [variable or 1]

**Existence and Uniqueness Theorem.** The initial value problem:

$\mathbf{x}' = \mathbf{f}(\mathbf{x}, t)$ with $\mathbf{x}(t_0) = \mathbf{x}_0$

has:

**Example 2.** Write the following 3rd–order initial value problem as a system of 1st–order differential equations.

$x''' + xx'' = \cos t$ with $x(0) = x'(0) = x''(0) = 0$

The general strategy is set $u_1 = x$, $u_2 = x'$, $u_3 = x''$, and so on if needed. This process shows that, if you can solve systems of [1st]–order differential equations, then you can actually solve [any]-order differential equations!

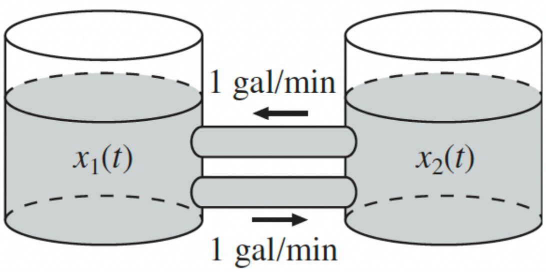
C. **Linear Systems of Differential Equations.** Consider the **linear** system:

$$\begin{cases} x' = tx \\ y' = -3x - 2y + 5\cos t \end{cases}$$

A **linear** system of differential equations can be put into **matrix form**:

and is **homogeneous** if:

**Example 3.** Consider two tanks, each containing 500 gallons of salt solution. Solution is pumped between the pipes connecting them at 1 gal/min. Let $x_1(t)$ and $x_2(t)$ represent the salt content in lbs of the tanks at time $t$ min. Initially: the first tank contains 5 lbs of salt and the second tank contains pure water.




Find a linear system describing the mixing process and write it in **matrix form**.

The basic principles are:

[salt rate of change] = [rate in] − [rate out]

and:

[rate in] = [inflow rate]·[salt concentration]

[rate out] = [outflow] · [salt concentration]

and:

$$[\text{salt concentration}] = \frac{[\text{total salt}]}{[\text{total volume}]}$$

Is the linear system homogeneous?