A. **Phase Plane.** The system:

$$\begin{cases} x' = -3x - y \\ y' = -x - 3y \end{cases}$$

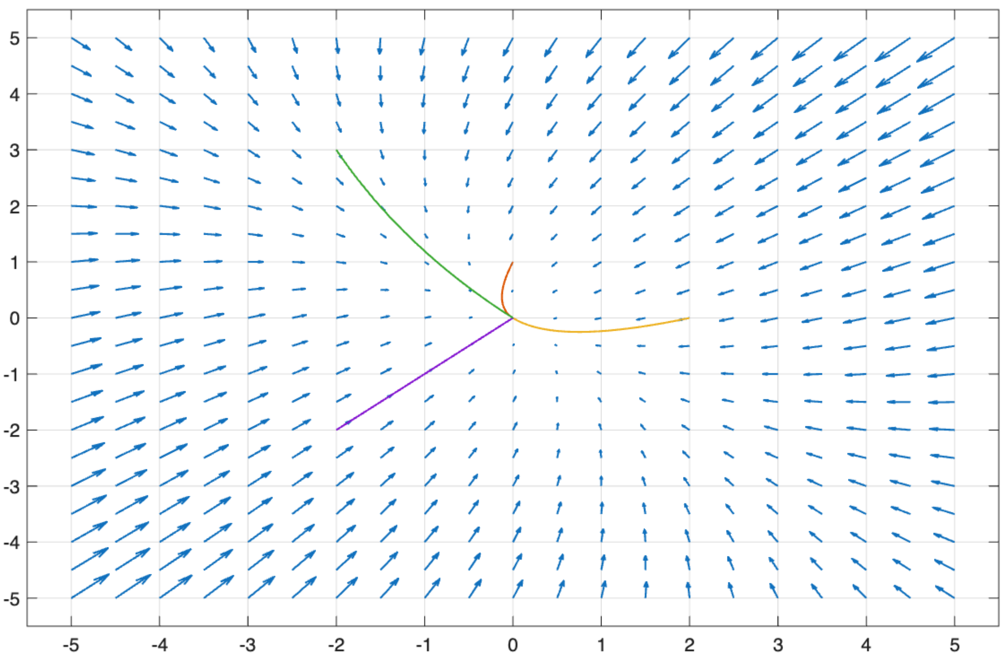
we solved is planar. So it has a direction field.

Remember: **planar** meant linear, autonomous, and 2D. To have a direction field in the xy-plane, all we need is autonomous (does not involve t) and 2D. In this case, the xy-plane was called the **phase plane**.

We had found solution:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{-2t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 e^{-4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Because the eigenvalues are negative, as $t \to \infty$ the solution "sinks" to the origin.



classification: **because eigenvalues are:**

Here are other examples for systems $\mathbf{x}' = A\mathbf{x}$ for different 2×2 matrices A.

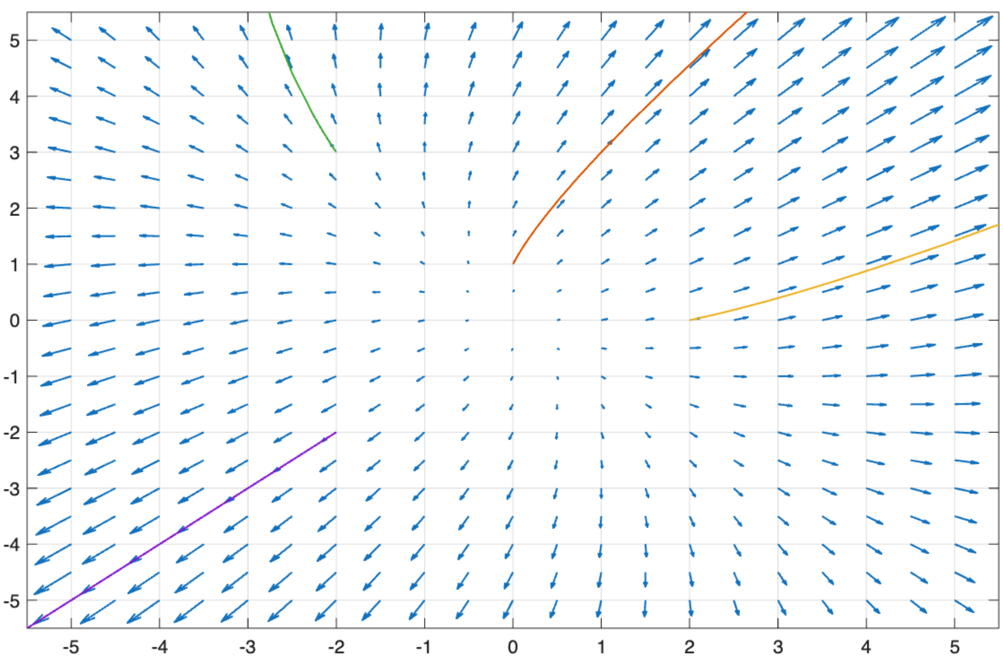
This is the direction field for $\mathbf{x}' = A\mathbf{x}$ for:

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

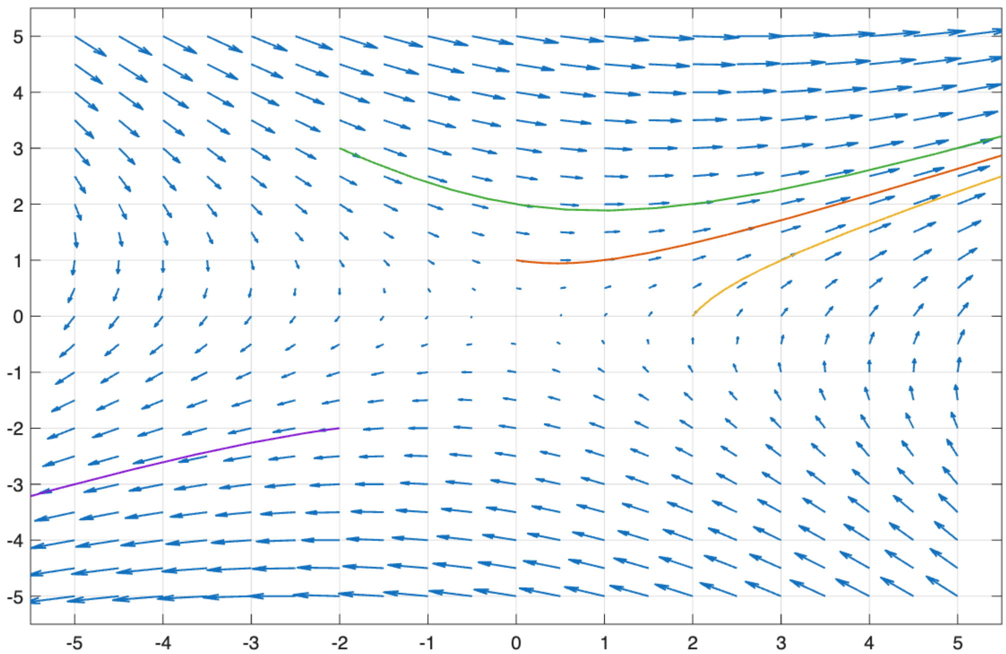
which has solution:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{2t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Because the eigenvalues are positive, as $t \to \infty$ the solutions go to ∞.



classification: **because eigenvalues are:**



This is the direction field for $\mathbf{x}' = A\mathbf{x}$ for:

$$A = \begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}$$

which has solution:

$$\mathbf{x} = C_1 e^{-3t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Because the eigenvalues are mixed sign, as t increases, the solutions both move towards and away from the origin.

classification: **because eigenvalues are:**

B. **Complex Eigenvalues.**

Let A be a 2×2 matrix with complex eigenvalue λ with complex eigenvector $\mathbf{v}$. Therefore a fundamental set of **complex** solutions to $\mathbf{x}' = A\mathbf{x}$ are:

$\mathbf{z} =$

$\overline{\mathbf{z}} =$

Remember that complex eigenstuff comes in complex conjugate pairs. So the conjugate $\overline{\lambda}$ is also a complex eigenvalue, and its complex eignvector is $\overline{\mathbf{v}}$.

In the scenario above, a **real** fundamental set of solutions to $\mathbf{x}' = A\mathbf{x}$ is:

$\mathbf{x}_1 =$

$\mathbf{x}_2 =$

and the general solution is a linear combination of these: $\mathbf{x} = C_1\mathbf{x}_1 + C_2\mathbf{x}_2$.

The idea is that, as mentioned in an earlier margin note, a linear combination of solutions to a **homogeneous** linear system is also a solution. In our case, we use that the real and imaginary parts of $\mathbf{z}$ are linear combinations of $\mathbf{z}$ and $\overline{\mathbf{z}}$. Namely:

$$\mathrm{Re}(\mathbf{z}) = \frac{\mathbf{z} + \overline{\mathbf{z}}}{2} \text{ and } \mathrm{Im}(\mathbf{z}) = \frac{\mathbf{z} - \overline{\mathbf{z}}}{2i}$$

Example 1. Find the general solution to $\mathbf{x}' = A\mathbf{x}$ where:

$$A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

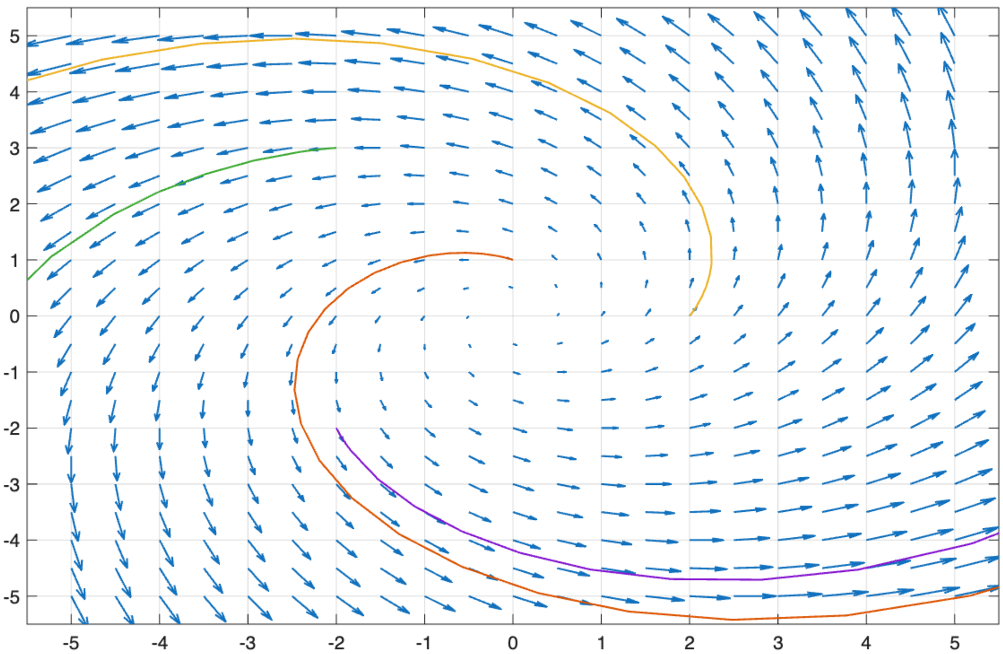
Recall:

$e^{i\theta} = \cos\theta + i\sin\theta$

C. **Phase Plane: Complex Eigenvalues.** We just solved system $\mathbf{x}' = A\mathbf{x}$ where:

$$A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

The direction field for this system is:



This is the direction field for $\mathbf{x}' = A\mathbf{x}$ for:

$$A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

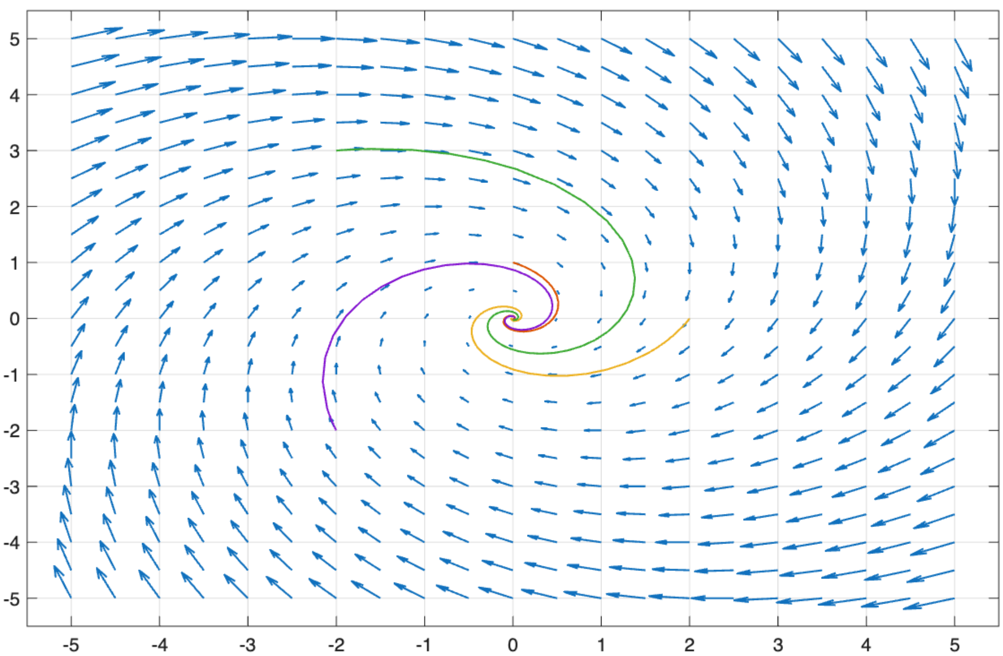
having eigenvalues $\lambda = 1 \pm 2i$ and:

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{1t} \begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix} + C_2 e^{1t} \begin{pmatrix} -\sin 2t \\ \cos 2t \end{pmatrix}$$

Because the real part of the eigenvalues is positive, and due to the cosine and sine, as $t \to \infty$ the solutions go to ∞ and circulate about the origin.

classification: **because the complex eigenvalues:**

Here are other examples where A has complex eigenvalues.



This is the direction field for $\mathbf{x}' = A\mathbf{x}$ for:

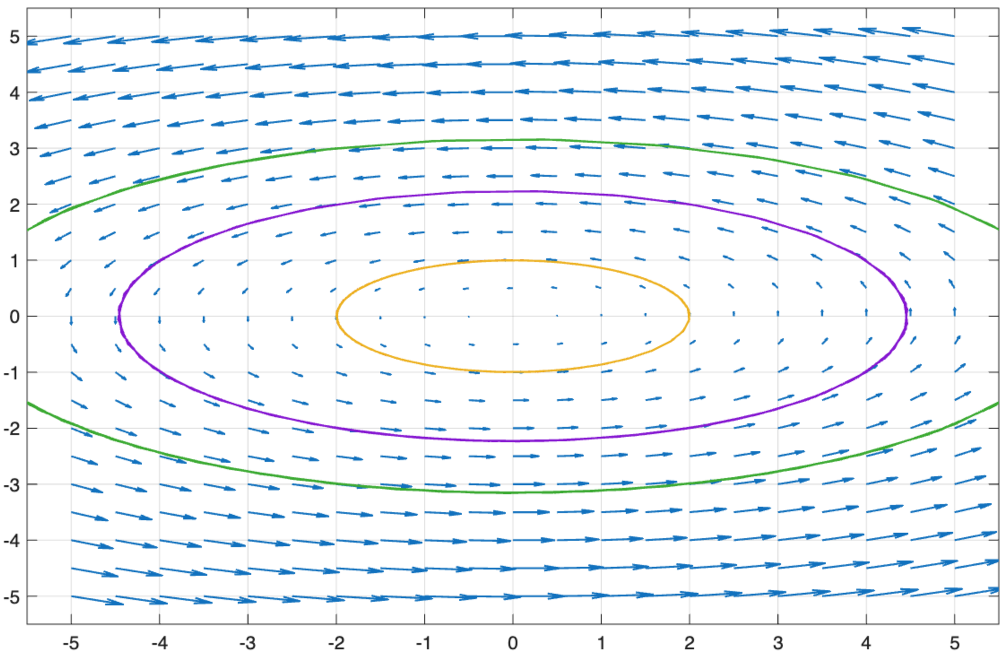
$$A = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix}$$

having eigenvalues $\lambda = -1 \pm 2i$ and:

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{-1t} \begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix} + C_2 e^{-1t} \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix}$$

Because the real part of the eigenvalues is negative, and due to the cosine and sine, as $t \to \infty$ the solutions aprooach the origin while circulating about it.

classification: **because the complex eigenvalues:**



This is the direction field for $\mathbf{x}' = A\mathbf{x}$ for:

$$A = \begin{pmatrix} 0 & -4 \\ 1 & 0 \end{pmatrix}$$

having eigenvalues $\lambda = \pm 2i$ and:

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 \begin{pmatrix} -2\sin 2t \\ \cos 2t \end{pmatrix} + C_2 \begin{pmatrix} 2\cos 2t \\ \sin 2t \end{pmatrix}$$

Because the real part of the eigenvalues is zero, and due to the cosine and sine, as $t \to \infty$ the solutions circle about the origin.

classification: **because the complex eigenvalues:**