A. **Undetermined Coefficients.** As you can see, variation of parameters can be quite the hassle. Let's chart a different tack. Consider again:

$\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t)$

We select a **trial solution** based on the form of the forcing term $\mathbf{f}$.

In the above scenario, if the forcing term has form:

$\mathbf{f}(t) = e^{ct}\mathbf{b}$ where $\mathbf{b}$ is a constant vector

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then, if $\lambda = c$ is **not**:

pick trial solution: $\mathbf{x}_p =$

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but, if $\lambda = c$ appears:

pick trial solution: $\mathbf{x}_p =$

The problem when $\lambda = c$ is an eigenvalue of A is that there are vectors of the form $e^{ct}\mathbf{v}$ that already solve the homogoneous equation $\mathbf{x}' = A\mathbf{x}$, and so have no chance of solving when there is a **nonzero** forcing term $\mathbf{f}$. The strategy we use here is very similar to when we applied undetermined coefficients to 2nd–order differential equations, where we multiplied by powers of t for repeated roots. It is not quite as simple here, because we are dealing with functions multiplied by **vectors**.

Example 1. Use undetermined coefficients to find a solution to $\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t)$ where:

$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ and $\mathbf{f}(t) = \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}$

Example 2. Use undetermined coefficients to find a solution to $\mathbf{x}' = A\mathbf{x}+\mathbf{f}$ where:

$A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$ and $\mathbf{f} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Solving the resulting system is a little tedious, so the steps have been executed in the margin.

Here is how you would finish solving the system after we removed the redundant equations. Our remaining equations can be rewritten with constants on the right:

I: $a_2 - b_2 = 0$

II: $2a_2 - a_1 + b_1 = 0$

III: $a_1 - a_0 + b_0 = 1$

IV: $b_1 - a_0 + b_0 = 0$

So converted to augmented matrix:

$$\begin{array}{cccccc|c} a_2 & b_2 & a_1 & b_1 & a_0 & b_0 & \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \end{array}$$

Next eliminate the pivot in the 2nd row by executing $\text{II} \mapsto \text{II} - 2\text{I}$

$$\begin{array}{cccccc|c} a_2 & b_2 & a_1 & b_1 & a_0 & b_0 & \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \end{array}$$

This is now in row–echelon form, where the pivots have been colored blue. The columns of a_0 and b_0 do not contain pivots, so those variables are free, meaning we can assign them any values. For simplicity: we pick $a_0 = b_0 = 0$. Our four equations become:

I: $a_2 - b_2 = 0$

II: $2b_2 - a_1 = 0$

III: $a_1 = 1$

IV: $b_1 = 0$

which we solve by back substitution to find $a_2 = b_2 = \frac{1}{2}$, $a_1 = 1$, $b_1 = 0$. We combine this with the earlier $a_0 = b_0 = 0$.