A. **Matrix Exponential.** If we could make sense of $e^{[\text{matrix}]}$ then:

$\mathbf{x}' = A\mathbf{x}$

would have immediate solution: $\mathbf{x} =$

Fortunately there is a way to make this make sense.

The series formula for the exponential is:

$e^t =$

If A is square, then the **matrix exponential** is defined by:

$e^A =$

and has the differentiation property:

$\frac{d}{dt}\left[e^{tA}\right] =$

and if 0 denotes a square zero matrix, has property:

$e^0 =$

Because e^A is an sum of matrices, it is itself a matrix.

Remember that a zero matrix is a matrix whose entries are all 0.

Example 1. Consider **diagonal** matrix:

A matrix is **diagonal** if its only nonzero entries are along the main diagonal from upper–left to bottom–right.

$$D = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

and find: e^D

Diagonal Exponential. If A is square and diagonal, with diagonal entries:

$$D = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

then:

$$e^{tD} =$$

Example 2. Consider matrix:

$$N = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

and find: e^N

A square matrix N is **nilpotent** if:

where k is called the **order** if it is minimal with this property.

In the 2×2 world every nonzero nilpotent matrix has order:

in which case: $e^{tN} =$

The fact about 2×2 nilpotent matrices is due to theoretical reasons we have not discussed. More generally, every $n \times n$ nilpotent matrix has order at most n.

B. **Exponential Properties.**

Product of Exponentials $\overset{?}{\mapsto}$ **Sum of Exponents.** Generally:

$e^A e^B$

However if A and B **commute**, meaning:

then it is the case that:

$e^A e^B$

The reason this property does not generally hold is because the matrix product is not commutative (commutative is about swapping the order of multiplication). Remember that the exponentials are themselves matrices. If it did hold, then:

$e^A e^B = e^{A+B} = e^{B+A} = e^B e^A$

which would imply e^A and e^B commute.

Let A be 2×2 matrix with only one eigenvalue λ that has been repeated.

Due to theoretical reasons:

$A = D + N$

where $D =$ is:

and: $N =$ is:

and D and N:

Use this to compute: $e^{tA} =$

That D and N commute is not complicated. Since $D = \lambda I$ is a multiple of the identity, D will commute with any matrix, because:

$DA = \lambda IA = \lambda A = \lambda AI = A\lambda I = AD$

where at some poine we have used the property that the scalars like λ commute with all matrix products.

If A is 2×2 with repeated eigenvalue λ then:

$e^{tA} =$

Example 3. Let:

$$A = \begin{pmatrix} -3 & -1 \\ 4 & 1 \end{pmatrix}$$

and find: e^{tA}

To save you time: $\det(A - \lambda I) = (\lambda + 1)^2$

Along the way, we double check that:

$DN = ND$

$N^2 = 0$

because, if you recall, this was supposed to hold for theoretical reasons.