Example 1. Use the exponential method, but without explicitly finding eigenvectors, to solve initial value problem $\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t)$ with $\mathbf{x}(0) = x_0$ where:

$$A = \begin{pmatrix} -4 & -1 \\ 1 & -2 \end{pmatrix} \text{ and } \mathbf{f}(t) = \begin{pmatrix} e^{-3t} \\ 0 \end{pmatrix} \text{ and } \mathbf{x}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

To save you time: $\det(A - \lambda I) = (\lambda + 3)^2$

This is possible, because the matrix in question is 2×2 with repeated eigenvalue $\lambda = c$. Remember, in this case we have the shortcut:

$e^{tA} = e^{ct}\,(I + tN)$ where $N = A - cI$

Example 2. Solve initial value problem $\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t)$ with $\mathbf{x}(0) = x_0$ where:

$$A = \begin{pmatrix} -2 & 1 \\ -5 & 2 \end{pmatrix} \text{ and } \mathbf{f} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } \mathbf{x}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

using the exponential method, and given that a fundamental matrix of A is:

$$M(t) = \begin{pmatrix} \sin t + 2\cos t & 2\sin t - \cos t \\ 5\cos t & 5\sin t \end{pmatrix}$$

Recall the formula for the exponential of A given a fundamental matrix is:

$e^{tA} = M(t)M(0)^{-1}$