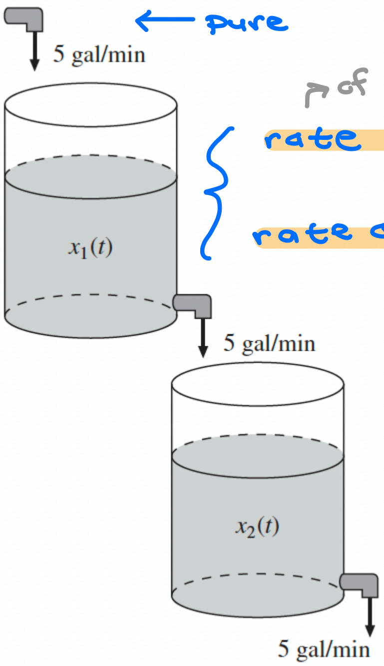
Example 1. Initially, the upper tank initially contains 200 gallons of salt solution with salt concentration of 0.2 lb/gal, and the lower tank also initially contains 200 gallon of salt solution, but with salt concentration 0.1 lb/gal. As time progresses, water flows between the tanks in the manner indicated, with **pure** water flowing into the upper tank. Find formulas for the salt contents $x_1(t)$ and $x_2(t)$ in the upper and lower tanks, in lbs.

$$\begin{cases} x_1(0) = 200 \text{ gal} \cdot \dfrac{0.2 \text{ lb}}{\text{gal}} = 40 \text{ lbs} \\[2ex] x_2(0) = 200 \text{ gal} \cdot \dfrac{0.1 \text{ lb}}{\text{gal}} = 20 \text{ lbs} \end{cases}$$



← pure

# of lbs of salt

$$\begin{cases} \text{rate in: } 0 \dfrac{\text{lbs}}{\text{min}} \\[2ex] \text{rate out: } \dfrac{5 \text{ gal}}{\text{min}} \cdot \dfrac{x_1 \text{ lbs}}{200 \text{ gal}} = \dfrac{x_1}{40} \dfrac{\text{lbs}}{\text{min}} \end{cases}$$

$$\begin{cases} \text{rate in: } \dfrac{x_1}{40} \dfrac{\text{lbs}}{\text{min}} \\[2ex] \text{rate out: } \dfrac{x_2}{40} \dfrac{\text{lbs}}{\text{min}} \end{cases}$$

Remember, if A is 2×2 with a repeated eigenvalue then we have the shortcut:

$e^{tA} = e^{ct}(I + tN)$ where $N = A - cI$

for us, $c = -\frac{1}{40}$ ($= \lambda$)

the rest

$$\underbrace{\boxed{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}'}}_{\vec{x}'} = \underbrace{\boxed{\begin{array}{cc} x_1 & x_2 \\ \hline \begin{pmatrix} -1/40 & 0 \\ 1/40 & -1/40 \end{pmatrix} \end{array}}}_{A} \underbrace{\boxed{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}}_{\vec{x}} + \underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{\text{homogeneous}}$$

$\det(A - \lambda I) = 0$

$\left(-\frac{1}{40} - \lambda\right)^2 = 0$

$\lambda = -\frac{1}{40}$, repeated

shortcut: $e^{tA} = \cdots = e^{-t/40} \begin{pmatrix} 1 & 0 \\ t/40 & 1 \end{pmatrix}$

$$\vec{x}(0) = \underbrace{\boxed{\begin{pmatrix} 40 \\ 20 \end{pmatrix}}}_{\vec{x}_0}$$

exp method for homogeneous: $\vec{x} = e^{tA} \vec{x}_0$

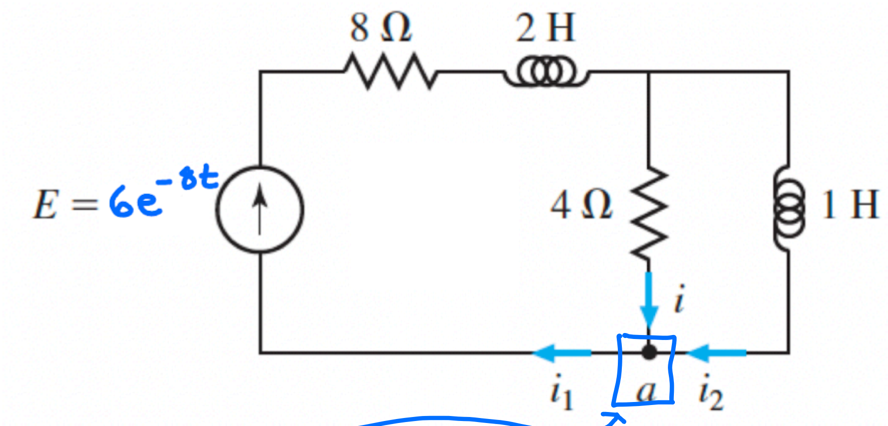
$$\vec{x} = e^{-t/40} \begin{pmatrix} 1 & 0 \\ t/40 & 1 \end{pmatrix} \begin{pmatrix} 40 \\ 20 \end{pmatrix}$$

$$= e^{-t/40} \begin{pmatrix} 40 \\ t + 20 \end{pmatrix}$$

Example 2. Find the state–free solution—i.e. with initial values equal 0–for currents i_1 and i_2 in the circuit:

$\hookrightarrow \vec{i}(0) = \vec{0}$

$\vec{i} = \begin{pmatrix} i_1 \\ i_2 \end{pmatrix}$



Some reminders about electrical circuits. The zigzags are resistors and the curlies are inductors.

Kirkhoff's Current Law:

- at any juncture, the current in equals the current out.

Kirkhoff's Voltage Law:

- directed sum of voltages around any closed loop equals 0

Impeding voltages across components:

- resistor: Ohm's Law $E_R = RI$
- capacitor: Capacitance Law $E_C = \frac{Q}{C}$
- inductor: Faraday's Law $E_L = L\frac{dI}{dt}$

Current is derivative of charge:

- $I = \frac{dQ}{dt}$

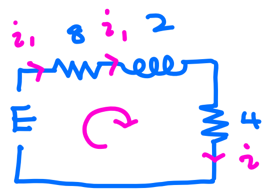
We apply Kirkhoff's voltage law to the left loop and the right loop.

current law: current in = current out

$$i + i_2 = i_1$$

$$i = i_1 - i_2$$

voltage law: left loop

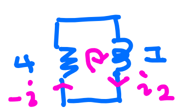


$$6e^{-8t} = 8i_1 + 2i_1' + \boxed{4i} \leftarrow 4i_1 - 4i_2$$

$\downarrow$ rearrange

$$i_1' = -6i_1 + 2i_2 + 3e^{-8t}$$

voltage law: right loop



$\hookrightarrow$ corrective neg so counterclockwise

$$0 = \boxed{4(-i)} + 1i_2' \qquad -4i_1 + 4i_2$$

$$i_2' = 4i_1 - 4i_2$$

$$\begin{pmatrix} i_1 \\ i_2 \end{pmatrix}' = \underbrace{\boxed{\begin{array}{cc} i_1 & i_2 \\ \begin{pmatrix} -6 & 2 \\ 4 & -4 \end{pmatrix} \end{array}}}_{\tau_A} \begin{pmatrix} i_1 \\ i_2 \end{pmatrix} + \overbrace{\boxed{\begin{pmatrix} 3e^{-8t} \\ 0 \end{pmatrix}}}^{\vec{f}}$$

state-free : $\vec{i}(0) = \vec{0}$

Example 2 Continued.

You should find:

$$A = \begin{pmatrix} -6 & 2 \\ 4 & -4 \end{pmatrix} \text{ and } \mathbf{f}(t) = \begin{pmatrix} 3e^{-8t} \\ 0 \end{pmatrix}$$

in which case it will turn out that:

$$e^{tA} = \frac{1}{3}\begin{pmatrix} e^{-2t} + 2e^{-8t} & e^{-2t} - e^{-8t} \\ 2e^{-2t} - 2e^{-8t} & 2e^{-2t} + e^{-8t} \end{pmatrix}$$

(state-free soln)

used this

pull in front of integral

$$3e^{-8t}\begin{pmatrix} e^{8u} \\ 0 \end{pmatrix}$$

$$\vec{z} = e^{tA} * \vec{f}(t)$$

$$= \frac{1}{3}\int_0^t \boxed{\begin{pmatrix} e^{-2u} + 2e^{-8u} & * \\ 2e^{-2u} - 2e^{-8u} & * \end{pmatrix}} \boxed{\begin{pmatrix} 3e^{-8(t-u)} \\ 0 \end{pmatrix}} du$$

$$= e^{-8t}\int_0^t \begin{pmatrix} e^{6u} + 2 \\ 2e^{6u} - 2 \end{pmatrix} du$$

$$= e^{-8t}\begin{pmatrix} \frac{1}{6}e^{6t} - \frac{1}{6} + 2t \\ \frac{1}{3}e^{6t} - \frac{1}{3} - 2t \end{pmatrix}$$

Bonus Formula: If A is 2×2 w/ complex evalues $\lambda = a \pm bi$, then:

$$e^{tA} = e^{at}\left(\cos(bt)I + \sin(bt)R\right) \quad \text{where} \quad R = \frac{1}{b}(A - aI)$$

Justification:

not obvious

Fact. $A = aI + bR$ where $R^2 = -I$

$$e^{tA} = e^{atI + btR}$$

$$= e^{atI}e^{btR}$$

$$= e^{at}\left(I + btR + \frac{b^2t^2\boxed{R^2}}{2!} + \frac{b^3t^3\boxed{R^3}}{3!} + \frac{b^4t^4\boxed{R^4}}{4!} + \dots \right)$$

$= -I$ $\qquad$ $= -R$ $\qquad$ $= I$

$$= e^{at}\left[\left(I - \frac{b^2t^2}{2!} + \frac{b^4t^4}{4!} + \dots\right)I + \left(bt - \frac{b^3t^3}{3!} + \frac{b^5t^5}{5!} + \dots\right)R\right]$$

$$= e^{at}\left[\cos bt \cdot I + \sin bt \cdot R\right]$$