A. **Exact Forms.** A **differential 1-form** is an expression:

$\omega = M(x, y)\,dx + N(x, y)\,dy$

For example: rewrite the following equation as a 1-form set equal to 0:

$$\frac{dy}{dx} = -\frac{2x + y}{x - 1}$$

Differential forms unify our approach to solving first–order differential equations. They can be used to address first–order separable and linear, but also to address others! The key to this unification is to write differential forms as **exact** forms, see below, using integrating factors as necessary.

The **total derivative** of a function $F(x, y)$ is:

$dF =$

F_x and F_y are shorthand for $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$.

For example:

$d(x^2 + y^2) =$

A differential form $\omega = M\,dx + N\,dy$ is **exact** if there exists a **potential** F so:

or in other words so:

$M =$

$N =$

For "nice" differential forms, you can check exactness by confirming:

(closed criterion)

A "nice" differential form here includes the case that M and N are defined and continuously differentiable everywhere. Continuously differentiable means they are differentiable, and their derivatives are continuous.

The reason that the closed criterion should hold is that, if $M = F_x$ and $N = F_y$, then $M_y = F_{xy}$ and $N_x = F_{yx}$. By Clairaut's Theorem (Calc III!) these mixed derivatives should be equal!

Example 1. Decide whether the form is exact, and if so locate a potential F.

(a) $(2x + y)\,dx + (xy)\,dy$

(b) $(2x + y)\,dx + (x - 1)\,dy$

The strategy to locate potential F for exact:

$M\,dx + N\,dy$

is as follows:

Step 1. Set up equations

I: $F_x = M$

II: $F_y = N$

Step 2. Integrate (I) with respect to dx.

I: $F = \int M\,dx$

When you compute this integral you need to add $+$ [constant with respect to x], i.e. $+\,g(y)$ where $g(y)$ is currently unknown.

Step 3. Plug the new equation I into equation II and solve for $g(y)$.

B. **Exact Equations.** An **exact equation** is one of form:

$M\,dx + N\,dy = 0$

where $\omega = M\,dx + N\,dy$ is exact.

Suppose F is a potential.

Exact Equation Solution If:

$M\,dx + N\,dy = 0$

is an exact equation with a potential F, then the solution satisfies:

Example 2. Solve the initial value problem:

$(2x + y) + (x - 1)y' = 0$ with $y(2) = 0$

by converting it to an exact equation.

In an earlier example you found a potential for the diferential form:

$(2x + y)\,dx + (x - 1)\,dy$

to be:

$F(x, y) = x^2 + xy - y$