A. **Counting.** Let us consider the experiment of rolling a fair 6–sided die.

The **sample space** Ω is the set of all possible outcomes. In the case of the die roll:

$\Omega =$

Generally: a **set** is a collection of objects, and we refer to those objects as **elements** of the set. So, the elements of the sample space are the outcomes.

Because the die is fair, each outcome is:

Consider the event that the die lands on an odd number. Formally, an **event** is a subset of the sample space. For the event of an odd number, that subset is:

$\Omega_{\text{odd}} =$

Since each outcome is equally likely, the **probability** of the event Ω_{odd} is:

$\mathbb{P}(\text{roll is odd}) = \mathbb{P}(\Omega_{\text{odd}}) =$

If S is a set, then $|S|$ denotes the size of that set, i.e. the number of elements it contains.

For equally likely outcomes, the key to computing probability is **counting**. So, for this topic, we concentrate on counting, and counting alone. We will re–introduce probability in the next topic.

Let us instead roll the 6–sided die two times. The **new** combined sample space for the first and second rolls is:

$$\Omega = \left\{ \begin{array}{c} (1,1),(1,2),(1,3),(1,4),(1,5),(1,6), \\ (2,1),(2,2),(2,3),(2,4),(2,5),(2,6), \\ (3,1),(3,2),(3,3),(3,4),(3,5),(3,6), \\ (4,1),(4,2),(4,3),(4,4),(4,5),(4,6), \\ (5,1),(5,2),(5,3),(5,4),(5,5),(5,6), \\ (6,1),(6,2),(6,3),(6,4),(6,5),(6,6) \end{array} \right\}$$

which has size $|\Omega| =$

Basic Principle of Counting. Suppose that Trial 1 and Trial 2 are performed. If Trial 1 results in m possible outcomes, and if, for each possible outcome of Trial 1, Trial 2 results in n possible outcomes, then the **combined** number of outcomes for Trial 1 and 2 is:

Example 1. An ID consists of 3 upper–case letters, each from A to Z, followed by 2 digits, each from 0 to 9. For example, ABC01 or ZZZ99.

(a) Count the number of passwords, where repetition of characters is allowed.

(b) Count the number of passwords, where repetition of characters is not allowed.

B. **Permutations.** A **permutation** of a set is a way of ordering its elements in a row. For example, all the possible permutations of $\{a, b, c\}$ are:

The number of permutations of a set of size 3 is thus:

$3! =$

Counting Permutations. The number of permutations of a set of size n is:

$n! =$

And we define $0! =$

The idea behind $0! = 1$ is that there is only 1 way to arrange a set consisting of no elements, do nothing!

A k**–permutation** of a set is a way of ordering k of its elements in a row. For example, all the possible 2–permutations of $\{a, b, c, d, e\}$ are:

ab, ba, ac, ca, ad, da, ae, ea, bc, cb, bd, db, be, eb, cd, dc, ce, ec, de, ed

Because this set has size 5, the number of 2–permutations is denoted by:

${}_5P_2 =$

Counting k–Permutations.

If $k \in \{0, \ldots, n\}$, then the number of k–permutations of a set of size n is:

${}_nP_k =$

If $k \notin \{0, \ldots, n\}$, then we define ${}_nP_k =$

The notation $\in$ is read as "in" or "belongs to".

For example, let us revisit the example of counting passwords consisting of 3 upper–case letters, followed by 2 digits.

repetition not allowed $\implies$

C. **Multiset Permutations.** A **multiset** is a set where repetition of its elements is allowed. Consider for example the multiset:

Note that plain sets do not account for repetition. For example, a set cannot have the element a twice. It either has the element a, or does not.

$\{a, a, a, b, b\}$

The **multiplicity** of an element is the number of times it appears. So, for the set above, a has multiplicity 3 and b has multiplicity 2. A permutation of a multiset is an ordering of its elements in a row. For example, the permutations of $\{a, a, a, b, b\}$ are:

$aaabb$, $aabab$, $abaab$, $baaab$, $aabba$, $ababa$, $baaba$, $abbaa$, $babaa$, $bbaaa$

We can alternatively count the number of these multiset permutations by **first** counting the number of regular set permutations of $\{a_1, a_2, a_3, b_1, b_2\}$ and then **dividing** by the **factors of overcounting**.

Multiset Permutations. The number of permutations of a multiset of size n, with r distinct elements having multiplicities $n_1, n_2, \ldots, n_r$, equals the **multinomial coefficient**:

$$\binom{n}{n_1, n_2, \ldots, n_r} =$$

This defines the multinomial coefficient so long as $n_1, \ldots, n_r$ are nonnegative integers and $n_1 + \cdots + n_r = n$. If either of these conditions fails, we define this multinomial coefficient to be 0.

Example 2. How many ways can the letters in the word PEPPER be arranged?