A. **Combinations.** A k**–combination** of a set is a subset of size k. For example, the 3–combinations of $\{a, b, c, d\}$ are:

Note that plain sets do not record order. Therefore $\{0, 1\}$ and $\{1, 0\}$ are the same set, because they contain the same elements.

We can count the number of k–combinations of a set of size n by counting its number of k–permutations and then dividing by the factor of overcounting.

In a k–permutation, order matters, while in a k–combination, order does not.

> k**–Combinations.** If $k \in \{0, \ldots, n\}$, then the number of k–combinations of a set of size n is the **binomial coefficient**:
>
> $$\binom{n}{k} =$$
>
> If $k \notin \{0, \ldots, n\}$, then we define $\binom{n}{k} =$

Note, it so happens that the binomial coefficient equals a specific multinomial coefficient:

$$\binom{n}{k} = \binom{n}{k, n-k}$$

This is because we can alternatively think of a k–combination as an ordering of the multiset $\{\mathrm{IN}, \ldots, \mathrm{IN}, \mathrm{OUT}, \ldots, \mathrm{OUT}\}$ of k IN's and $n-k$ OUT's, where an IN indicates an element is in the subset, while an OUT indicates it is not in the subset.

Example 1. Give a combinatorial argument (no computations) justifying the following identities.

Symmetry Identity: $\binom{n}{k} = \binom{n}{n-k}$

Pascal's Rule: $\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$

Example 2. A committee of 4 people is to be formed from 20 people.

(a) How many ways are there to do this, if everybody on the commitee has the same role?

(b) How many ways are there to do this, if the committee has 4 distinct roles, and each committee member takes on exactly one role.

(c) How many ways are there to do this, if the 20 people are divided into 5 distinct committees of exactly 4 people each, and on a given committee, everybody has the same role?

Divisions Into Distinct Groups of Specified Sizes. The number of ways to divide n objects into r distinct groups of sizes $n_1, n_2, \ldots, n_r$ is:

So in addition to the interpretation involving counting multisets, we can use this interpretation.

B. **Multinomial Theorem.** Multinomial coefficients help expand powers of sums.

$$(x_1+x_2+\cdots+x_r)^n = \underbrace{(x_1+x_2+\cdots+x_r)(x_1+x_2+\cdots+x_r)\cdots(x_1+x_2+\cdots+x_r)}_{n \text{ times}}$$

You can get a specific term in the expanded expression as follow: choose the number n_1 of factors to "pick" x_1 from, the number n_2 of remaining factors to "pick" x_2 from, ..., and the remaining number n_r of factors to "pick" x_r from. Each choice of factors in this specific manner will give you monomial:

A monomial is a product of variables.

$$\text{where } n_1+n_2+\cdots+n_r =$$

The **number** of ways of choosing factors in this specific manner is the number of ways of **dividing** the n factors into r distinct groups with size n_1 (the x_1 group), size n_2 (the x_2 group), through size n_r (the x_r group). This is:

Together, we obtain:

Multinomial Theorem.

$$(x_1+x_2+\cdots+x_r)^n = \sum_{\substack{n_1+n_2+\cdots+n_r=n \\ n_1,n_2,\ldots,n_r \geqslant 0}}$$

In other words, there is a term in the sum for each $(n_1,\ldots,n_r)$ so that $n_1+\cdots+n_r=n$ and $n_1,n_2,\ldots,n_r \geqslant 0$, and that term is the one specified.

As a special case:

$$(x+y)^n =$$

Giving us the more familiar:

Binomial Theorem.

$$(x+y)^n = \sum_{k=0}^{n}$$

Example 3. Use the binomial theorem to prove the following identity.

Binomial Sum Identity: $\sum_{k=0}^{n} \binom{n}{k} =$

This can also be proven combinatorially. The sum is the number of subsets (of all possible sizes) of a size–n set, since each $\binom{n}{k}$ measures the number of subsets of specific size k. The number of all subsets can also be counted by thinking of choosing them through a process: for each element of the size–n set, make one of 2 choices, either to include it or not include it in your subset. The total number of such choices is 2^n.

Example 4. Find the coefficient of x^3y^2 in the expansion of:

$(2 + 3x + 4y)^{100}$