Example 1. A bag contains 6 white and 5 black balls. We draw exactly 3 of them. What is the probability that exactly one of the drawn balls is white?

$\{W_1, W_2, W_3, W_4, W_5, W_6, B_1, B_2, B_3, B_4, B_5\}$

Approach 1. Draw them in order: one at a time, without replacement.

$\Omega = \{3\text{-permutations}\}$ ← equally likely outcomes

↳ $\underset{1}{\underline{W_5}}\quad \underset{2}{\underline{B_4}}\quad \underset{3}{\underline{B_3}}$

$|\Omega| = {}_{11}P_3 = 11 \cdot 10 \cdot 9$

$E = \{3\text{-permutations w/ exactly 1 } W\}$

$|E| = \underbrace{\underline{3}}_{\text{location of } W} \cdot \underbrace{\underline{6}}_{\text{which } W} \cdot \underbrace{\underline{{}_5P_2}}_{\text{draw 2 blacks}} = 3 \cdot 6 \cdot 5 \cdot 4$

$\mathbb{P}(E) = \dfrac{|E|}{|\Omega|} = \dfrac{\cancel{3} \cdot \cancel{6} \cdot \cancel{5} \cdot 4}{11 \cdot \cancel{10} \cdot \cancel{9}} = \dfrac{4}{11}$

Approach 2. Draw them all at once: ignoring the order of drawing.

$\Omega = \{3\text{-combinations}\}$ ← equally likely outcomes

$|\Omega| = \binom{11}{3} = 11 \cdot 5 \cdot 3$

$E = \{3\text{-comb'n's w/ exactly 1 } W\}$

$|E| = \underbrace{\binom{6}{1}}_{1\,W} \underbrace{\binom{5}{2}}_{2\,B\text{'s}} = 6 \cdot 5 \cdot 2$

$\mathbb{P}(E) = \dfrac{{}^{2}\cancel{6} \cdot \cancel{5} \cdot 2}{11 \cdot \cancel{5} \cdot \cancel{3}} = \dfrac{4}{11}$

This problem models that you can feel free to adjust the sample space containing the event, as long as the outcomes in the new sample space are equally likely.

This can be subtle. If you went back to the rolls of a pair of fair 6–sided dice, the unordered sample space actually does **not** have equally likely outcomes. For example, the probability of rolling a pair of 4's is $\frac{1}{36}$, as it comes from the singular ordered outcome $(4,4)$, while the probability of rolling a 3 and 5, in any order, is $\frac{1}{18}$, as it comes from two ordered outcomes $(3,5)$ and $(5,3)$.

Example 2. In the game of bridge, the entire deck of 52 cards is dealt to 4 players, 13 cards each.

(a) What is the probability one player gets all 13 spades?

E_i = [event player i gets all 13 spades]

in

Ω_i = {unordered hands for player i}

$$\mathbb{P}(E_i) = \frac{|E_i|}{|\Omega_i|} = \frac{1}{\binom{52}{13}}$$

$$\mathbb{P}\left(\bigsqcup_{i=1}^{4} E_i\right) = \sum_{i=1}^{4} \mathbb{P}(E_i) = \frac{4}{\binom{52}{13}}$$

(b) What is the probability each player receives 1 ace?

Here we will take as our sample space the set of unordered hands of cards for the 4 distinct players. We could also achieve the same result by using a sample space consisting of ordered hands. Or, you could achieve the same result by using unordered hands to indistinct players.

$$|\Omega| = \binom{52}{13,13,13,13} \leftarrow$$ equally likely outcomes

A = [event of 1 A for each player]

$$|A| = \underbrace{\binom{4}{1,1,1,1}}_{\text{distribute aces to players}} \cdot \underbrace{\binom{48}{12,12,12,12}}_{\text{distribute remaining cards}}$$

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|}$$

A. **Derangements.** Suppose that n people throw their hats into the center of a room, then the hats are mixed and returned to people uniformly at random.

What is the probability that no one gets their own hat back?

A **derangement** is a permutation in which no object ends up in its initial spot. So, in this problem, we are calculating the proability of a derangement.

$\Omega = \{\text{permutations}\}$

$\hookrightarrow \frac{2}{1}\ \frac{1}{2}\ \frac{3}{3}\ \cdots\ \frac{5}{n}$

$|\Omega| = n!$

$E_i = [\text{event person } i \text{ gets their hat back}]$

we want: $1 - \mathbb{P}\left(\bigcup_{i=1}^{n} E_i\right)$

inclusion-exclusion:

$$\mathbb{P}\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{r=1}^{n} (-1)^{r+1} \sum_{\substack{r\text{-wise} \\ \text{intersections}}} \mathbb{P}(E_{i_1}E_{i_2}\cdots E_{i_r})$$

- $|E_{i_1}E_{i_2}\cdots E_{i_r}| = \underbrace{1}_{\substack{\text{persons} \\ i_1,\dots,i_r \\ \text{get hat back}}} \cdot \underbrace{(n-r)!}_{\substack{\text{distribute} \\ \text{remaining} \\ \text{hats}}}$

- $\mathbb{P}(E_{i_1}\cdots E_{i_r}) = \dfrac{(n-r)!}{n!}$

- $\displaystyle\sum_{\substack{r\text{-wise} \\ \text{intersections}}} \mathbb{P}(E_{i_1}\cdots E_{i_r}) = \binom{n}{r} \cdot \frac{(n-r)!}{n!} = \frac{1}{r!}$

$\hookrightarrow \#\ \text{terms} = \binom{n}{r}$

Note that as $n \to \infty$ the probability of a derangement approaches:

$$\sum_{r=0}^{\infty} \frac{(-1)^r}{r!} = e^{-1} = \frac{1}{e}$$

The reason for this is tied to Poisson random variables, which we will discuss later. Feel free to come back and read this note again, after you have learned about them. For each person, the probability that they get their hat back is $\frac{1}{n}$. When n is large, the events of each person getting their hat back become roughly independent, and thus the number of success conists of enormous number n (roughly) independent trials, each with miniscule success rate $1/n$. The expected number of hats returned is $n \cdot \frac{1}{n} = 1$, and so this is the situation in which a Poisson variable with paramter $\lambda = n \cdot \frac{1}{n} = 1$ applies. In this case, the formula for k successes is given by $e^{-1}/k!$. So, for no successes, is e^{-1}.

$\Lsh$ new :o just prettily writing answer

so: $\mathbb{P}\left(\bigcup_{i=1}^{n} E_i\right) = \displaystyle\sum_{r=1}^{n} (-1)^{r+1} \cdot \frac{1}{r!}$

$$1 - \mathbb{P}\left(\bigcup_{i=1}^{n} E_i\right) = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^n}{n!}$$

$$= \sum_{r=0}^{n} \frac{(-1)^r}{r!}$$

$\leftarrow$ so pretty