A. **Conditional Probability.** For example, let $\Omega = \{\text{population}\}$.

$H = [\text{event of heart disease}] \longrightarrow \mathbb{P}(H) = 5\%$

$O = [\text{event of over 75 years old}] \longrightarrow \mathbb{P}(O) = 7\%$

O stands for old.

$HO = [\text{heart disease and over 75 years old}] \longrightarrow \mathbb{P}(HO) = 1.8\%$

Given that your patient is over 75 years old, what is the probability they have heart disease.

Yes, you are a doctor now. Congratulations.

$\mathbb{P}(\text{heart disease given over 75 years old}) = \mathbb{P}(H \mid O)$

Conditional Probability. If E and F are events, and F occurs with positive probability, then the **conditional probability** of E given F is:

$\mathbb{P}(E \mid F) =$

If the probability of event F is zero, the conditional probability is treated as finite but undefined, basically meaning in practice that, if you multiply the conditional probability by 0, even if the conditional probability is undefined, you may treat the result as 0.

You can think of the conditional probability as what happens if you shrink you sample space Ω to be the event F. In the example above, we shrunk from the general population, to those people who are really old.

The ordinary probability space utilizes the probability function $\mathbb{P}(\,\cdot\,)$. However, if we have observed event F, then the way we compute probability changes: our new probability function is $\mathbb{P}(\,\cdot \mid F\,)$ which we call a **conditioned probability**. Importantly, this new probability function defines a probability space, meaning all the usual axioms are satisfied.

Conditioned Probability Function. If F is an event that occurs with positive probability, then the **conditioned probability** given F is $\mathbb{P}(\,\cdot \mid F\,)$. This defines the **conditioned probability space** given F, which is in fact a probability space.

See the textbook for a proof that it is a probability space, i.e. that it satisfies all the axioms.

Example 1. Joe lost his key. He thinks the key is in his pants with probability $\frac{1}{3}$, in his jacket with probability $\frac{1}{2}$, and elsewhere with probability $\frac{1}{6}$. Joe just checked his jacket, and did not find the key. What is the probability that the key is in his pants?

B. **Multiplication Rule.** For events E and F we have:

$$\mathbb{P}(E \mid F) = \frac{\mathbb{P}(EF)}{\mathbb{P}(F)} \implies \mathbb{P}(EF) =$$

We can iterate this equality to obtain an identity for the probability of the intersection of multiple events.

Multiplication Rule. For events $E_1, E_2, \ldots, E_n$, we have:

$$\mathbb{P}(E_1 E_2 \cdots E_n) =$$

Example 2. Recall the matching hats problem: Suppose that n people throw their hats into the center of a room, then the hats are mixed and returned to people uniformly at random.

We had calculated the probability that no one gets their hat back is:

$$\mathbb{P}(n\text{–derangement}) = \sum_{r=0}^{n} \frac{(-1)^r}{r!}$$

What is the probability that exactly k people get their hat back?

Recall: a **derangement** is a permutation where no object ends up back in its original place.

C. **Law of Total Probability.** For events E and F, since $\Omega = F \sqcup F^c$, we have:

$E =$

Therefore, if the event F has positive probability:

$\mathbb{P}(E) =$

Law of Total Probability.

If $\Omega = F_1 \sqcup F_2 \sqcup \cdots$ expresses the sample space as a countable **disjoint** union, or, in other words, if the events F_i are mutually exclusive and their total probability is 1, then:

$\mathbb{P}(E) =$

As an important special case, for an event F with positive probability:

$\mathbb{P}(E) = \mathbb{P}(E \mid F)\mathbb{P}(F) + \mathbb{P}(E \mid F^c)\mathbb{P}(F^c)$

Example 3. A blood test detects a certain disease, when present, with probability 95%. However, the test gives a "false positive" with probability 1%. In the population, 0.5% of people have the disease.

What is the probability a randomly selected person from the population tests positive?