A. **Bayes' Formula.** We can attempt to compute a conditional probability by reversing the roles of conditioning. For events $E$ and $F$ we have:

$\mathbb{P}(E \mid F) =$

> **Bayes' Formula.** If $E$ and $F$ are events, and $F$ has positive probability, then:
>
> $\mathbb{P}(E \mid F) =$

This simple formula leads to many counterintuitive results, because in practice, it explicitly accounts for the easily overlooked ways that the "information" granted by $F$ affects the balance of $E$ and $E^c$.

**Example 1.** Return to the earlier setup: A blood test detects a certain disease, when present, with probability $95\%$. However, the test gives a "false positive" with probability $1\%$. In the population, $0.5\%$ of people have the disease.

Given that a person tests positive, what it the probability that they have the disease?

Do you see what I mean about Bayes formula leading to counterintuitive results? $95\%$ accuracy when the disease is present, and $99\%$ accuracy when the disease is not present sound great. But, if you test positive, there is only a roughly $32\%$ chance you actually have the disease?! The counterintuitivity comes from the easily overlooked ways that having testing positive the balance of having and not having the disease. The percent of the population with TRUE positives is $0.95 \cdot 0.5\% = 0.475\%$, while the percent of the population with FALSE positives is $0.01 \cdot 99.5\% = 0.995\%$. Because there are SO many people who do not have the disease, just by sheer number, the proportion of FALSE positives is high.

**Example 2.** A box has 3 cards.

- one card has 2 red sides
- one card has 2 black sides
- one card has 1 red and 1 black side

You pick a card uniformly at random, and observe one side to be red. What is the probability the other side is red?

B. **Independence.** Two events $E$ and $F$ are **independent**, intuitively, if knowing $F$ does not affect the probability of $E$, and vice versa. That is:

$\mathbb{P}(E \mid F) =$

$\mathbb{P}(F \mid E) =$

**Independence.** Two events $E$ and $F$ are **independent**, written $E \perp F$, if:

$\mathbb{P}(EF) =$

In addition, each of the following is equivalent to $E$ and $F$ being independent:

- $\mathbb{P}(E \mid F) = \mathbb{P}(E)$ (assuming $F$ has positive probability)
- $\mathbb{P}(F \mid E) = \mathbb{P}(F)$ (assuming $E$ has positive probability)
- $E$ and $F^c$ are independent
- $E^c$ and $F$ are independent
- $E^c$ and $F^c$ are independent

The intuition behind the last three bullets is that, if knowing $F$ does not change the probability that $E$ occurs, then it also does not change the probability that $E$ does **not** occur.

**Example 3.** We toss two fair dice.

- Let $\Sigma_k$ be the event that the sum is $k$.
- Let $F_i$ be the event that the first die has value $i$.
- Let $S_j$ be the event that the second die has value $j$.

(a) Are $\Sigma_6$ and $F_4$ independent?

(b) Are $\Sigma_7$ and $F_4$ independent? How about $\Sigma_7$ and $S_4$? And $F_4$ and $S_4$?

(c) Are $\Sigma_7$ and $F_4S_4$ independent?

This example poses two warnings.

First, knowing that $\Sigma_7 \perp F_4$ and $\Sigma_7 \perp S_4$ did **not** guarantee that $\Sigma_7 \perp F_4S_4$.

Second, knowing that events are pairwise–independent, i.e. any pair of them is independent, does **not** guarantee they are $3$–wise–independent, in the sense that:

$$\mathbb{P}(\Sigma_7F_4S_4) \neq \mathbb{P}(\Sigma_7)\mathbb{P}(F_4)\mathbb{P}(S_4)$$

C. **Independence of Multiple Events.**

We say 3 events $E$, $F$, and $G$ are **independent** if both:

- they are pairwise–independent, meaning:

  $\mathbb{P}(EF) =$

  $\mathbb{P}(FG) =$

  $\mathbb{P}(EG) =$

- **and** they are 3–wise–independent, meaning:

  $\mathbb{P}(EFG) =$

Pairwise–independent does not mean the same thing as independent. Part c of the previous example showed the insufficiency of pairwise indepedndence.

**Independence of Multiple Events.** Events $E_1, E_2, \ldots, E_n$ are said to be **independent** if and only if, for any $1 \leqslant i_1 < i_2 < \cdots i_r \leqslant n$, we have:

$\mathbb{P}(E_{i_1} E_{i_2} \cdots E_{i_r}) =$

That is, they are $r$–wise independent, for any $r$.