A. **Geometric Random Variables.** Recall the sample space Ω of sequences of independent coin tosses, with $\mathbb{P}(H) = p$ and $\mathbb{P}(T) = 1 - p$.

Let X be the number of tosses until the first heads. For example:

Sequence of Tosses	Value of X	probability
H		
TH		
TTH		
$\underbrace{TT\cdots T}_{k-1 \text{ tails}}H$		

Geometric Random Variable. Let Ω be the sample space of sequences of independent trials, each with success rate p, and failure rate $1 - p$.

Then $X : \Omega \to \mathbb{N}$, defined by:

$X(\text{sequence}) = \text{number of trials until first success}$

is the **geometric random variable** with parameter n and p:

$X \sim \textbf{Geom}(p)$

Its probability mass function is given by, assuming $k \geqslant 1$:

$p(k) = \mathbb{P}(X = k) = \mathbb{P}(k \text{ trials until first success}) =$

B. **Uniform Random Variables.** Let $\Omega = \{a_1, \ldots, a_n\}$ be the probability space of **equally likely** selections from from the set $\{a_1, \ldots, a_n\}$. Let $X : \Omega \to \Omega$ be the value of the selection.

$\mathbb{P}(X = a_i) =$

Uniform Random Variable. Let $\Omega = \{a_1, \ldots, a_n\}$ be the probability space of **equally likely** selections from the set $\{a_1, \ldots, a_n\}$. Let $X : \Omega \to \Omega$ be the value of the selection.

Then X is the **uniform random variable** on $\{a_1, \ldots, a_n\}$:

$X \sim \textbf{Uniform}(\{a_1, \ldots, a_n\})$

Its probability mass function defined by:

$p(a_i) = \mathbb{P}(X = a_i) =$

C. **Poisson Random Variables.** The binomial random variable $N \sim \text{Binom}(n, p)$ counts the number of successes after n indpendent trials, each with probability of success p. Intuitively:

[average number of success] $= \lambda =$

We will formalize this notion of average number of success in topic A5.

So we can write:

$N \sim \text{Binom}(n, p) =$

Suppose the number of trials approaches ∞, but the average λ stays constant:

$$\lim_{n\to\infty} \mathbb{P}(N = k) = \lim_{n\to\infty} \binom{n}{k} p^k (1-p)^{n-k}$$

One of the ways of defining e^x is:

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$$

Poisson Random Variable. A binomial random variable with an extremely large number of trials, relative to a small average number λ of successes, can be approximated by a **Poisson random variable**:

$X \sim \textbf{Poisson}(\lambda)$

which has probability mass function given by, assuming $k \geqslant 0$:

$\mathbb{P}(\text{exactly } k \text{ successes}) \approx p(k) = \mathbb{P}(X = k) =$

Practically, the probability computations for Poisson random variables $\text{Poisson}(\lambda)$ are far simpler than those for binomial random variables $\text{Binom}(n, \frac{\lambda}{n})$, when n is extremely large.

Example 1. Each minute on Route 110, the probability of a car crash is $\frac{1}{120}$, independently of all other minutes. Let N be the number of crashes in 4 hours. Approximate $\mathbb{P}(N = 3)$.

The number of "trials" is the number of minutes in 4 hours, which is extremely large, compared to the average number of crashes, which is:

$\lambda = 4 \text{ hours} \cdot \frac{1}{120} \frac{\text{crashes}}{\text{minute}} = 2 \text{ crashes}$

D. **Sums of Independent Poisson Random Variables.** Let $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu)$ be **independent** Poisson random variables, and let's investigate their sum.

$\mathbb{P}(X + Y = n) =$

Sum of Independent Poisson. If $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu)$ are **independent** Poisson random variables, then:

$X + Y \sim$

Example 2. Let X and Y be the number of returning and new customers that come to a store over the course of a day. Assume that X and Y are independent Poisson random variables, and that, on average, 10 returning and 20 new customers come into the store over the course of a day. Let Z be the total number of customers that come to the store. Find $\mathbb{P}(Z = 32)$.