A. **Hypergeometric Random Variables.** Suppose a bag contains N balls, with m being white, and $N - m$ being black. You draw n balls uniformly at random.

Consider the function $X : \Omega \to \mathbb{N}$ that assigns, to a drawing of n balls, the number of **white** balls drawn.

$\mathbb{P}(X = k) = \mathbb{P}(\text{exactly } k \text{ white balls drawn}) =$

Hypergeometric Random Variable. Let Ω be the sample space of equally likely selections of n objects from N, where m objects are **special**.

Then $X : \Omega \to \mathbb{N}$, defined by:

$X(\text{selection}) = \text{number of } \textbf{special} \text{ objects selected}$

is the **hypergeometric random variable** with parameter N and m:

$X \sim \textbf{HyperGeom}(N, m, n)$

Its probability mass function is:

$p(k) = \mathbb{P}(X = k) = \mathbb{P}(\text{exactly } k \textbf{ special} \text{ objects selected}) =$

B. **Expectation.** Let X be the value a roll of a fair 6–sided die. Then X is uniform on $\{1, 2, 3, 4, 5, 6\}$. The **average** or **expected** value of the roll is:

$\mathbb{E}[X] =$

Expectation. Let X be a random variable. Then the **expected value** of X is:

$\mathbb{E}[X] =$

The idea is you **weight** each value by the probability that value is achieved.

C. **Expectation of a Discrete Uniform Random Variable.** Let X be a uniform random variable on $\{1, 2, \dots, n\}$. Then:

$\mathbb{E}[X] =$

You will want the formula:

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

Uniform Expectation. If X is uniform on $\{1, 2, \dots, n\}$ then:

$\mathbb{E}[X] =$

D. **Expectation of a Bernoulli Random Variable.** Let X be a Bernoulli random variable with parameter p. Then:

$\mathbb{E}[X] =$

Bernoulli Expectation. If $X \sim \text{Bernoulli}(p)$ then:

$\mathbb{E}[X] =$

As a special case, if $\mathbb{1}_E$ is the indicator function of an event E then:

$\mathbb{E}[\mathbb{1}_E] =$

E. **Expectation of a Poisson Random Variable.** Let X be a Poisson random variable with parameter λ, for example the number of car crashes over a week. Recall, in this case, λ is supposed to represent the **average** number of car crashes in that week. But that would suggest $\mathbb{E}[X] = \lambda$. Let's confirm:

$\mathbb{E}[X] =$

Recall: the probability mass function of $\text{Poisson}(\lambda)$ is:

$$p(k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

if $k \in \mathbb{N}$, and is 0 otherwise.

Recall the Taylor series:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

and its derivative:

$$e^x = \sum_{k=0}^{\infty} \frac{kx^{k-1}}{k!}$$

Poisson Expectation. If $X \sim \text{Poisson}(\lambda)$ then:

$\mathbb{E}[X] =$

F. **Linearity of Expectation.** Let Ω be the sample space of flips of 2 fair coins, with one side labelled 0 and the other side labelled 1. Let X be the value of the first flip, and Y be the value of the flip, and $Z = 2X + 2Y$ be twice the value of the sum. Let's rewrite the definition of expectation using singular outcomes:

$\mathbb{E}[Z] =$

Expectation Using Singular Outcomes. Let X be a discrete random variable on sample space X. An equivalent definition of expectation is:

$$\mathbb{E}[X] = \sum_{\omega \in \Omega}$$

Returning to the example of coin flips, where $Z = 2X + 2Y$, we can also rearrange the above calculation to see that $\mathbb{E}[Z] = 2\mathbb{E}[X] + 2\mathbb{E}[Y]$.

The following results are none other than this kind of rearranging, along with factoring out constants.

Expectation is Linear. Let X and Y be random variables on a common probability space, and α and β be constants. Then:

$\mathbb{E}[\alpha X + \beta Y] =$

This can be extended to n random variables:

$$\mathbb{E}[\alpha_1 X_1 + \cdots + \alpha_n X_n] = \alpha_1 \mathbb{E}[X_1] + \cdots + \alpha_n \mathbb{E}[X_n]$$

The linearity of expectation is extremely nice, as the probability mass functions themselves do not behave linearly.

G. **Expectation of a Binomial Random Variable.** Let X be a binomial random variable, counting the number of success after n independent trials of an experiment, each with success probability p. We had noted that X is a sum of independent Bernoulli random variables:

$X =$

and consequently:

$\mathbb{E}[X] =$

Recall that the expectation of $\mathrm{Bernoulli}(p)$ is p.

Binomial Expectation. If $X \sim \mathrm{Binom}(n, p)$ then:

$\mathbb{E}[X] =$

This makes sense, because after n trials with success rate p, shouldn't the average number of successes just be np?

Example 1. Recall the matching hats problem: Suppose that n people throw their hats into the center of a room, then the hats are mixed and returned to people uniformly at random. Let X be the number of people who get their own hat back.

In an earlier example, we showed that if E_i is the event that the ith person gets their hat back, then:

$$\mathbb{P}(E_i) = \frac{1}{n}$$

$$X = \mathbb{1}_{E_1} + \mathbb{1}_{E_2} + \cdots + \mathbb{1}_{E_n}$$

Recall, an indicator function is equivalent to a Bernoulli random variable with probability of success equal to the probability of the event being indicated.

and thus:

$\mathbb{E}[X] =$