Example 1. Let X be uniform on $(0, 1)$ and let $Y = e^X$. Find the expected value of Y by first finding its cdf and pdf.

Later: we will see two other, better, methods for solving this problem.

We begin by recalling the cdf of a uniform random variable on $(0, 1)$ to be:

$\mathbb{P}(X \leqslant x) = x$ for $x \in (0, 1)$

And next we find the cdf of $Y = e^X$.

A. **Tail Formula for Expectation.** There is another, often convenient, formula for the expectation, that applies to **nonnegative** random variables.

To motivate it, let's look at a **discrete** random variable $X : \Omega \to \mathbb{N}$ with probability mass function $p(k) = \mathbb{P}(X = k)$ first. Then:

$$\begin{aligned}\mathbb{E}[X] = 0 \cdot p(0) + 1 \cdot p(1) + 2 \cdot p(2) + 3 \cdot p(3) + \cdots = p(1) + p(2) + p(3) + \cdots \\ + p(2) + p(3) + \cdots \\ + p(3) + \cdots \\ + \cdots \\ \vdots\end{aligned}$$

And thus, in this case where X has values in $\mathbb{N}$, we have:

$$\mathbb{E}[X] = \sum_{k=0}^{\infty}$$

A similar rearranging can be done in the continuous setting; it just requires changing the order of integrals instead of the orders of sums. Doing so yields:

Tail Formula for Expectation.

Let $X \geqslant 0$ be a **nonnegative** continuous random variable. Then:

$$\mathbb{E}[X] =$$

The function $\mathbb{P}(X > x)$ is called the **tail distribution function** (tdf) and equals 1 minus the cumulative distribution function.

Example 2. Let X be uniform on $(0,1)$ and $Y = e^X$. Use the tail formula for expectation to find $\mathbb{E}[Y]$.

We already solved this problem a different way. In doing so we found:

if $y \in \text{support}(Y) = (1, e)$ then $\mathbb{P}(Y \leqslant y) = \ln y$

Recall:

$\int \ln x \, dx = x \ln x - x + C$

B. **Expectation of a Function of the Random Variable.** Recall that for a discrete random variable X, we found that:

$$\mathbb{E}[g(X)] = \sum_x g(x) \cdot \mathbb{P}(X = x)$$

For continuous random variables, this looks like:

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot \mathbb{P}(x \leqslant X \leqslant x + dx) =$$

Expectation of Funcion of a Random Variables. If $g(X)$ is a function of a continuous random variable X, with pdf $f_X(x)$, then:

$$\mathbb{E}[g(X)] =$$

Example 3. Let X be uniform on $(0, 1)$ and let $Y = e^X$. Find $\mathbb{E}[Y]$ by using the formula for expectation of a function $g(X)$ of a random variable.

This is the third time we have encountered this problem. The method we apply here is the fastest.

We begin by recalling the pdf of a uniform random variable on $(0, 1)$ to be:

$f_X(x) = 1$ for $x \in (0, 1)$