A. **Variance.** We define variance in the same way as we did for discrete random variables.

Variance of Continuous Random Variables. The **variance** of a continuous random variable X with expected value $\mathbb{E}[X] = \mu$ is:

$\mathbb{E}[(X-\mu)^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$

Several properties of expectation for discrete random variables carry over for continuous random variables.

Independent Sum. If X, Y are independent, then $\mathrm{Var}(X+Y) = \mathrm{Var}(X)+\mathrm{Var}(Y)$.

B. **Variance: Continuous Uniform.** Let X be uniform on (a, b). We had found its probability density function to be:

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$$

first moment: $\mathbb{E}[X] = \text{midpoint} = \dfrac{a+b}{2}$

second moment: $\mathbb{E}[X^2] =$

difference of cubes:

$b^3 - a^3 = (b-a)(a^2+ab+b^2)$

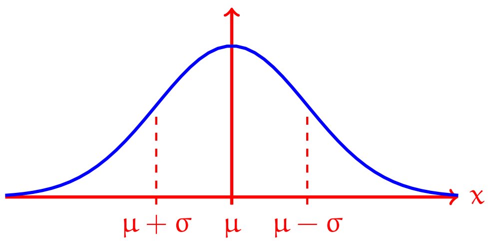
Variance: Continuous Uniform. If X is uniform on (a, b), then:

$\mathrm{Var}(X) =$

C. **Standard Normal Distribution.** The standard "bell curve" is the following:

As we will **prove** much later in the course, this shape inevitably arises when we look at an average of **many** independent trials of an experiment, which is why it is used so often, like assigning curves for letter grades, as it gives us an expected way for the distribution of exam scores to turn out.

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$



mean: $\mu = 0$

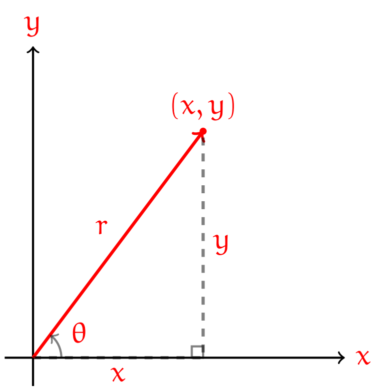
standard deviation: $\sigma = 1$

Let's check that $f(x)$ is a valid probability density function for a random variable X by confirming the total probability is 1, i.e. by confirming:

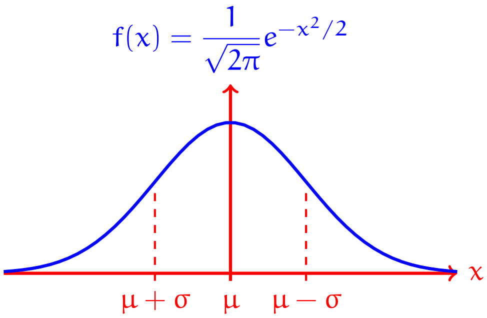
Standard Normal Random Variable: Normalizing Factor.

$$\int_{-\infty}^{\infty} e^{-x^2/2}\, dx =$$

It involves a fun and ingenious argument using **polar** coordinates.



Let X be the **standard normal random variable**, whose probability density function is the standard bell curve. In the sketch, the mean is given as $\mu = 0$ and the standard deviation as $\sigma = 1$, meaning the variance is $\sigma^2 = 1$. Let us confirm this:



mean: $\mu = 0$

standard deviation: $\sigma = 1$

Standard Normal Random Variable. The **standard normal random variable** X has probability density function:

$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ with mean $\mu = 0$ and standard deviation $\sigma = 1$

Its cumulative distribution function cannot be expressed using elementary functions, and is denoted by "Phi":

$\Phi(x) = \mathbb{P}(X \leqslant x)$

A table of values, taken from the texbook, is on the next page.

To use it, it helps to have the symmetry property: $\Phi(-x) =$

Table 5.1 Area $\Phi(x)$ Under the Standard Normal Curve to the Left of X.

X	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Example 1. Let X be the standard normal random variable, and find:

$\mathbb{P}(-1.1 \leqslant X \leqslant 0.2) =$