A. **Exponential Random Variables.** Consider a situation involving an enormous number of independent "trials", each with a very low probability of success. For example, car crashes, with each minute being a "trial". As we had seen, we can approximate the number of car crashes, say, per month, to be a **Poisson** random variable, with parameter λ being the expected number of car crashes per month.

Recall: a Poisson variable was a limit of binomial random variables, as we let the number of trials approach ∞, but maintain the mean λ.

Now consider the car crashes as an "arival process" of successes, with the 1st car crash considered the 1st "arrival", the 2nd car crash considered as the 2nd "arrival", and so on. Informally, we call this a **Poisson process**, an idea that we will formalize much later in the course.

Now, let T_1 be the time until the 1st arrival, and T_2 be the time after the 1st arrival until the 2nd arrival, and T_3 be the time after the 2nd arrival until the 3rd arrival, and so on. These times are random variables, called the **inter–arrival times**. Because time is continuous, these are **continuous** random variables.

As we will see when we formalize Poisson processes later in the course, the interarrival times are **independent and identically distributed**. The distribution is determined by the **exponential random variable**, with parameter λ equal to the expected number of car crashes (arrivals) per month (unit of time).

Recall: identically distributed means they have the same cdf, hence the same distribution of probability.

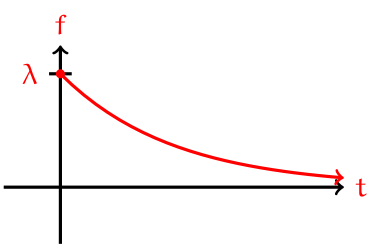
Exponential Random Variable. Consider a **Poisson process** of arrivals, with a **rate** of λ arrivals per unit of time.

Then the **inter–arrival times** are **iid** continous random variables, each with the distribution of the **exponential random variable** T with parameter λ, which is written:

$T = \text{Exp}(\lambda)$

and has pdf:

$$f(t) = \begin{cases} & \text{if } t < 0 \\ & \text{if } t \geqslant 0 \end{cases}$$



and tdf:

$$\mathbb{P}(T > t) = \begin{cases} & \text{if } t < 0 \\ & \text{if } t \geqslant 0 \end{cases}$$

Poisson processes will be formalized at the end of the course. For now think of "arrivals" that occur independently and could occur at any time.

Recall: iid stand for independent and identically distributed.

I use t here since because we are talking about time.

Recall that tdf stands for tail distribution function. It is 1 minus the cdf.

We calculate the tdf below. If $t \geqslant 0$:

$\mathbb{P}(T > t) =$

B. **Memoryless Property.** Let T be the time, in years, until a light burns out, and assume T is exponential with average lifetime of 2 years.

You can think of T as the arrival time until the first "success" of a Poisson process, where the process could be continued by replacing the light bulb.

Suppose the bulb has not gone out in the first 2 years. What is the probability that the bulb lasts at least an **additional** t years? In other words:

So, it is as if the waiting time timer until the bulb going out resets after year 2, if the bulb has not already gone out by that time. This "resetting" of the timer is called the **memoryless** property of the exponential random variable.

Memoryless Property.

Let T be an **exponential** random variable and $s, t \geqslant 0$. Then:

$\mathbb{P}(T > s + t \mid T > s) =$

C. **Gamma Function.** Let T be an exponential random variable with parameter λ. Then $T \geqslant 0$ and its tdf for $t \geqslant 0$ is:

$\mathbb{P}(T > t) = e^{-\lambda t}$

Since $T \geqslant 0$, we can use the tail formula for expectation:

This formula said that, if $X \geqslant 0$, then:

$$\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X > x)\ dx$$

first moment: $\mathbb{E}[T] =$

What if we want to compute higher moments? To do this, we introduce an ultra–special function called the **gamma function**.

$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\ dt$

Using the exact calculation above, but with $\lambda = 1$, we have:

$\Gamma(1) =$

Next, assuming $\alpha > 1$, let us apply integration by parts to:

Recall, integration by parts says:

$\int u\, dv = uv - \int v\, du$

$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\ dt$

So: $\Gamma(2) =$

$\Gamma(3) =$

$\Gamma(4) =$

The gamma function generalizes the factorial!

Gamma Function. The **gamma function** is defined by:

$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\ dt$

and satisfies:

functional equation: if $\alpha > 1$ then: $\Gamma(\alpha) =$

factorial property: if n is a positive integer, then: $\Gamma(n) =$

D. **Exponential Moments.** Let's use the gamma function to compute exponential moments. Let T be an exponential random variable with parameter λ. Then $T \geqslant 0$ and its pdf for $t \geqslant 0$ is:

$f(t) = \lambda e^{-\lambda t}$

nth moment: $\mathbb{E}[T^n] =$

Exponential Moments. Let $T \sim \mathrm{Exp}(\lambda)$. Its nth moment is:

$\mathbb{E}[T^n] =$

From which we can obtain:

$\mathbb{E}[T] =$

$\mathrm{Var}(T) =$

In terms of arrival processes, if the rate of arrivals per unit of time is λ, then the average time it takes for an arrival is $\frac{1}{\lambda}$, which sort of makes sense.