A. **Jointly Continuous Random Variables.** Now let's discuss the analogue to **joint mass functions** for **continuous random variables** X and Y.

First, let's revisit the visualization of a **joint mass function** (discrete) in Desmos and then transition to visualization of a **joint density function** (continuous) also in Desmos.

Jointly Continuous. Let X and Y be **continuous** random variables on a common probability space. We say that they are **jointly continuous** if they have an "integrable" **joint density function** $f(x,y)$. This means that, for any "measurable" region D in the xy–plane

joint density function = **jdf**

"Integrable" and "measurable" are technical terms, that you do not need to know. Intuitively, "integrable" means that the function can be integrated, and "measurable subset" means we can integrate over that subset.

$\mathbb{P}((X,Y) \in D) =$

Some consequences include:

Total Mass. $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx =$

Marginal cdfs.

$F_X(x) = \mathbb{P}(X \leqslant x) =$

$F_Y(y) = \mathbb{P}(Y \leqslant y) =$

Marginal pdfs. If $f_X(x)$ and $f_Y(Y)$ are the pdfs of X and Y, then:

$f_X(x) \overset{\text{a.e.}}{=}$

$f_Y(y) \overset{\text{a.e.}}{=}$

"a.e." stands for "almost everywhere" and is a tehnical term, that you do not need to know. Intuitively, it means that that they have the same integrals over any given "measurable" subset.

Joint cdf. The **joint cumulative distribution function** is defined as:

$F(x,y) = \mathbb{P}(X \leqslant x, Y \leqslant y) =$

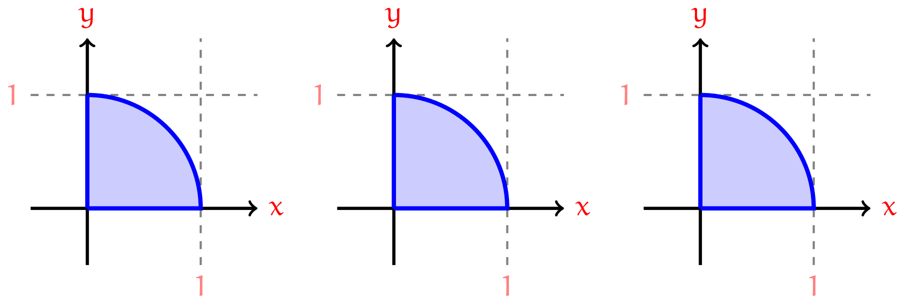
in which case: $f(x,y) \overset{\text{a.e.}}{=}$

This gives us a useful **heuristic**, i.e. informal rule for developing intuition:

$$f(x,y) \approx \frac{\mathbb{P}\left((X,Y) \in \qquad\qquad\qquad\right)}{\Delta x \Delta y} \longleftarrow \frac{\text{probability}}{\text{area}} \longleftarrow$$

Example 1. X and Y are jointly continuous with joint density function:

$$f(x,y) = \begin{cases} cxy & \text{if } x^2+y^2 \leqslant 1 \text{ and } x \geqslant 0 \text{ and } y \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$



(a) Find the value of c by integrating, as practice, in **polar** coordinates.

Polar coordinates involve r (distance from origin) and θ (counterclockwise rotation of position vector from positive x–axis) and are defined by:

$x = r\cos\theta$ and $y = r\sin\theta$

in which case, integrating using them requres:

$dA = r\, dr d\theta$

(b) Find the marginal density functions $f_X(x)$ and $f_Y(y)$.

B. **Independence and Joint Density Functions.** In B1, we said that two random variables (discrete or continuous) were **independent** if, for all x and y:

$$= \mathbb{P}(X \leqslant x, Y \leqslant y) = \mathbb{P}(X \leqslant x) \cdot \mathbb{P}(Y \leqslant y) =$$

or, in other words, if the joint cdf equals the product of the marginal cdfs. Assuming X and Y are **jointly continuous**, we take partial derivatives to phrase this in terms of density functions.

Recall that, if $F(x, y)$ is the joint cdf, then:

$$\frac{\partial^2 F}{\partial x \partial y} \overset{\text{a.e.}}{=} f(x, y)$$

equals the joint pdf, "almost everywhere".

Independence and Density Functions. Let X and Y be **jointly continuous** random variables, with jdf $f(x, y)$ and marginal pdf's $f_X(x)$ and $f_Y(y)$.

Then X and Y are **independent** if and only if:

$f(x, y) \overset{\text{a.e.}}{=}$

Example 2. Use the given joint density function of X and Y to decide if $X \perp Y$.

$$f(x,y) = \begin{cases} 4xy & \text{if } 0 \leqslant x \leqslant 1 \text{ and } 0 \leqslant y \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$