Example 1. Let X and Y be jointly continuous with jdf and joint support:

$$f(x, y) = 8xy \text{ if } x^2 + y^2 \leqslant 1 \text{ and } x \geqslant 0 \text{ and } y \geqslant 0$$

In a previous example, we had found marginal density functions and supports:

$$f_X(x) = 4x(1 - x^2) \text{ if } 0 \leqslant x \leqslant 1$$

$$f_Y(y) = 4y(1 - y^2) \text{ if } 0 \leqslant y \leqslant 1$$

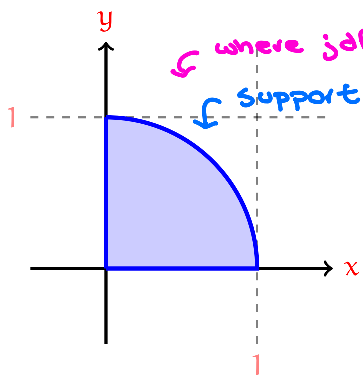
Are X and Y independent?

[prod. of marginal pdf's] $\overset{?}{=}$ [jdf]

$$4x(1-x^2) \cdot 4y(1-y^2) \neq 8xy$$

so: $X \not\perp Y$

Instead, come to the same conclusion, using only the shape of the **supports**.

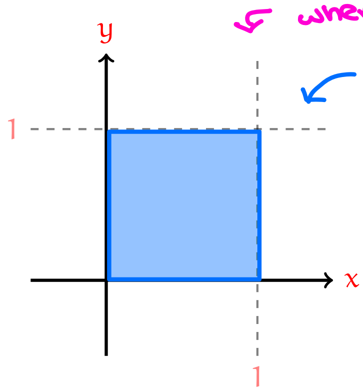


where jdf ≠ 0

support (X, Y)

where prod. of marginals ≠ 0

support(X) × support(Y)



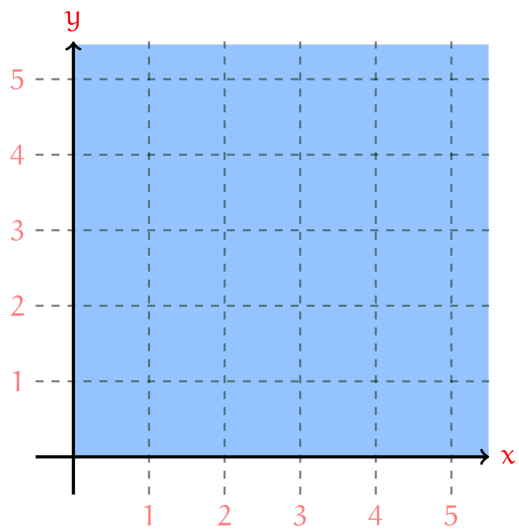
remark: $\perp$ rand. var's always have rectangles as joint support

This example shows that detecting indendence is not as simple as looking at formulas for the jdf in a piecewise expression, and noting that they have form function(x) · function(y). The shape of the **support**, i.e. where the probability distribution has nonzero probability, matters too. If the support is a horizontal–vertical **rectangle**, then it is okay to just check whether the jdf has the form function(x) · function(y), but usually, it is not.

Example 2. A light fixture holds two lightbulbs. The lifetimes X and Y of the two lightbulbs are **independent** exponential random variables, with the first lightbulb having life expectancy of $\frac{1}{5}$ decades, and the second lightbulb having life expectancy of $\frac{1}{3}$ decades.

Recall the pdf $\mathrm{Exp}(\lambda)$ is $\lambda e^{-\lambda t}$ for $t > 0$.

(a) Find the joint density function of X and Y.



$X \sim \mathrm{Exp}\left(\lambda = \frac{1}{1/5} = 5 \ \frac{\text{bulbs}}{\text{decade}}\right)$

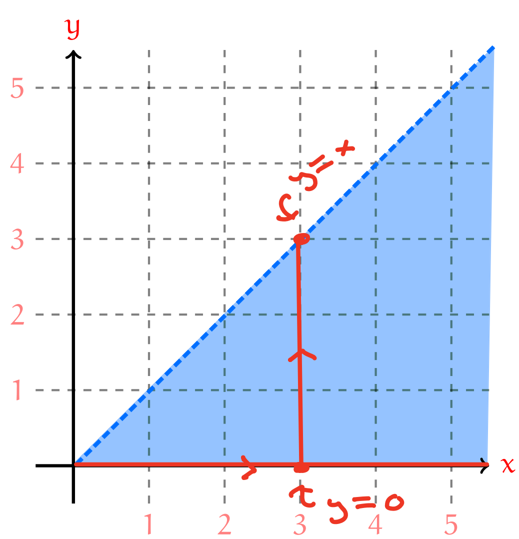
$Y \sim \mathrm{Exp}\left(\lambda = \frac{1}{1/3} = 3 \ \frac{\text{bulbs}}{\text{decade}}\right)$

b/c $X \perp Y$:

$$\text{support}(X, Y) = \text{support}(X) \times \text{support}(Y) = (0, \infty) \times (0, \infty)$$

$$\text{jdf}: f(x,y) = f_X(x)\, f_Y(y) = 5e^{-5x} \cdot 3e^{-3y} = 15e^{-(5x+3y)}$$

(b) Find the probability that the Y lightbulb burns out first.



$$P(Y < X) = \int_0^\infty \int_0^x 15e^{-(5x+3y)}\, dy\, dx$$

$$\begin{cases} u = 5x + 3y \rightarrow du = 3\,dy \\ y = 0 \rightarrow u = 5x \\ y = x \rightarrow u = 8x \end{cases}$$

$$= \int_0^\infty \int_{5x}^{8x} 5e^{-u}\, du\, dx$$

$$= \int_0^\infty 5e^{-5x} - 5e^{-8x}\, dx$$

$$= \int_0^\infty e^{-u}\, du - \int_0^\infty \frac{5}{8} e^{-u}\, du$$

$$= 1 - \frac{5}{8}$$

$$= \frac{3}{8} = 37.5\%$$

(c) After the X lightbulb burns out, it will take you $Z \sim \text{Uniform}(0,1)$ decades to change the lightbulb. Assume Z is **independent** from X and Y. What is the probability that the X lightbulb burns out, and is changed, all before the Y lightbulb burns out?

When setting up a triple integral in order $dy\,dx\,dz$, you use:

- full range of z
- full range of x, given y
- full range of y, given x and z

note: $f_Z(z) = 1$ for $0 < z < 1$

b/c X, Y, Z are ind.:

jdf: $f(x,y,z) = f_X(x)\, f_Y(y)\, f_Z(z)$

$= 15e^{-(5x+3y)}$

$\rightarrow$ w/ support: $x>0,\ y>0,\ 0<z<1$

$$P(X+Z<Y) = \int_0^1 \int_0^\infty \int_{x+z}^\infty 15e^{-(5x+3y)}\, dy\, dx\, dz$$

$$\begin{cases} u = 5x+3y \rightarrow du = 3\,dy \\ y = x+z \rightarrow u = 8x+3z \\ y = \infty \rightarrow u = \infty \end{cases}$$

$$= \int_0^1 \int_0^\infty \int_{8x+3z}^\infty 5e^{-u}\, du\, dx\, dz$$

$$= \int_0^1 \int_0^\infty 5e^{-(8x+3z)}\, dx\, dz$$

$$\begin{cases} u = 8x+3z \rightarrow du = 8\,dx \\ x = 0 \rightarrow u = 3z \\ x = \infty \rightarrow u = \infty \end{cases}$$

$$= \int_0^1 \int_{3z}^\infty \frac{5}{8} e^{-u}\, du\, dz$$

$$= \int_0^1 \frac{5}{8} e^{-3z}\, dz$$

$$\begin{cases} u = 3z \rightarrow du = 3\,dz \\ z = 0 \rightarrow u = 0 \\ z = 1 \rightarrow u = 3 \end{cases}$$

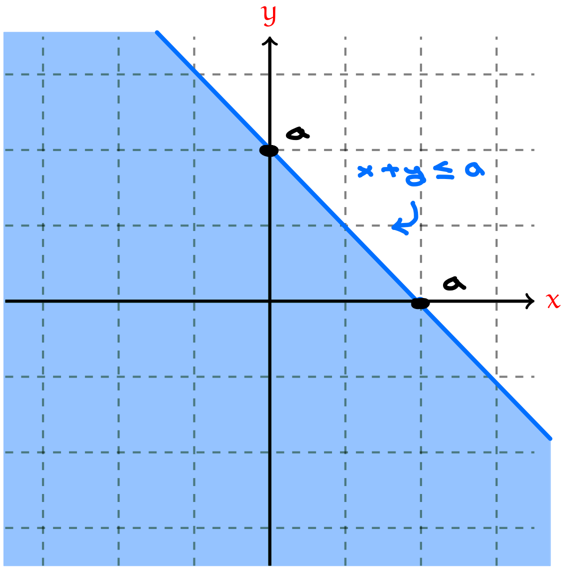
$$= \int_0^3 \frac{5}{24} e^{-u}\, du$$

$$= \frac{5}{24}(1 - e^{-3})$$

$\approx 19.8\%$

A. **Sums of Independent Random Variables.** Let X and Y be **jointly continuous** and **independent** random variables on a probability space. Let's find the cdf of their sum: $Z = X + Y$.

$$F_Z(a) = \mathbb{P}(Z \leqslant a) = \mathbb{P}(X + Y \leq a) = \iint_{x+y \leq a} f_X(x) f_Y(y)\, dA$$



sub: $z = x + y \to x = z - y$

$$= \int_{-\infty}^{a} \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)\, dy\, dz$$

And then let's find the pdf:

$$f_Z(a) = F_Z'(a) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)\, dy$$

We call this expression the **convolution** of f_X and f_Y and write it $f_X * f_Y$.

In general, the convolution of two functions $f(x)$ and $g(y)$ is:

$$(f * g)(z) = \int_{-\infty}^{\infty} f(z-y)g(y)\, dy$$

Independent Sum. Let X and Y be **jointly continuous** and **independent** random variables on a common probability space. Then the pdf of their sum $Z = X + Y$ is the **convolution** of the pdfs of X and Y:

$$f_Z(z) = (f_X * f_Y)(z) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)\, dy$$