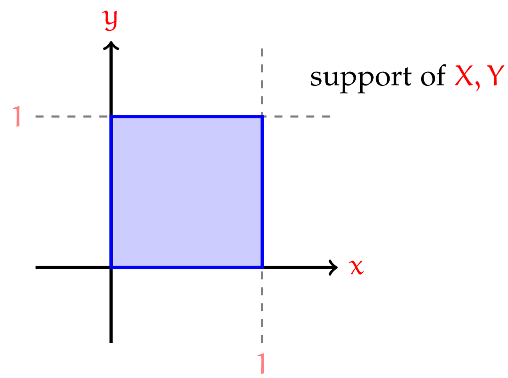
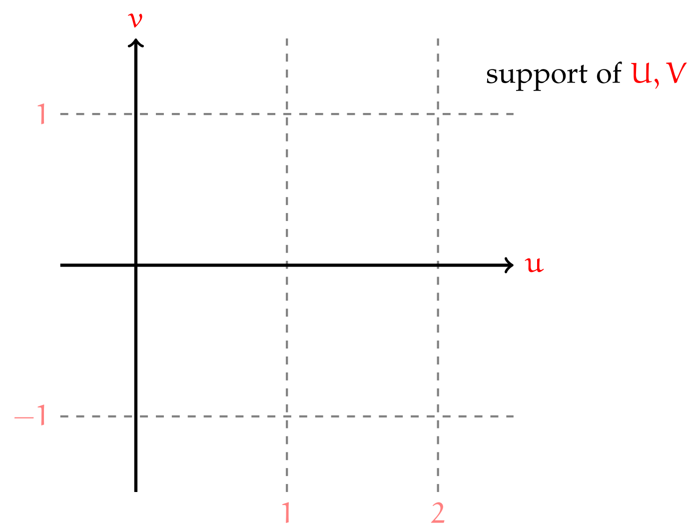
Example 1. Let X and Y be independent and uniform on $(0, 1)$. Let:

$U = X + Y$

$V = X - Y$

and find the joint density function of U and V. Sketch the **support** of the joint distribution of U and V, i.e. where the joint distribution has nonzero probability.





Example 2. Let R and $\Theta \sim \text{Uniform}(0, 2\pi)$ be **independent** random variables, where the pdf of R is given by:

$$f_R(r) = re^{-r^2/2}$$

Let:

$$X = R\cos\Theta$$

$$Y = R\sin\Theta$$

Show that X and Y are **independent standard normal** random variables.

If X and Y are independent and standard normal, we say that (X, Y) has the **standard bivariate normal** distribution.

We use the relationships between polar and standard coordinates:

$$r^2 = x^2 + y^2$$

$$\tan\theta = \frac{y}{x}$$

This example tells us that if the coordinates (X, Y) are have a standard bivariate normal distribution in the XY–plane, then the polar angle Θ is distributed uniformly, and the polar radius R is distributed exponentially with parameter $\lambda = \frac{1}{2}$. In a discussion problem, you will see that R can be written in terms of a uniform random variable, and thus, demonstrate how to write the standard bivariate normal distribution in terms of uniform distributions!

Recall earlier that we said a Cauchy distribution (rotating lamp) could be interpreted as a ratio of independent standard normal random variables? This example provides the evidence, because:

$$\frac{Y}{X} = \tan\Theta$$

and $\tan\Theta$ was precisely how we defined the Cauchy distribution.