A. **Joint Distributions and Expectation.** We can use the joint distribution of random variables X and Y to compute expectation of functions of X and Y, for example $\mathbb{E}[XY]$.

If X and Y are **jointly continuous** then:

$\mathbb{E}[g(X,Y)] =$

If X and Y are **discrete** then:

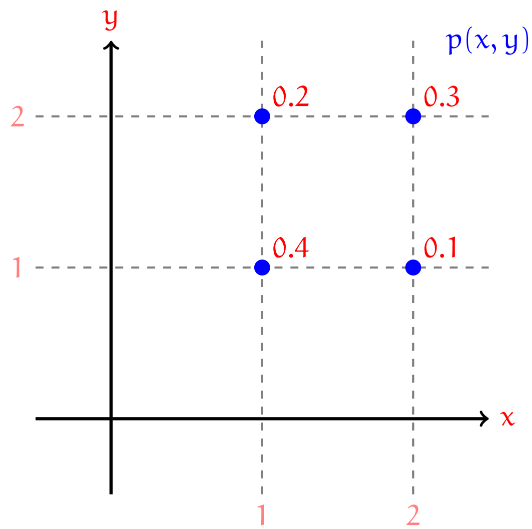
$\mathbb{E}[g(X,Y)] =$

Remember, the joint mass function is:

$$p(x,y) = \mathbb{P}(X = x, Y = y)$$

Example 1. Let $p(x, y)$ be a joint mass function whose values on its **support** and suppose that $p(1, 1) = 0.4$, $p(1, 2) = 0.2$, $p(2, 1) = 0.1$, $p(2, 2) = 0.3$.

Recall the support is where the random variable has nonzero probability.



Find: $\mathbb{E}[X] =$

$\mathbb{E}[Y] =$

$\mathbb{E}[X^2] =$

$\mathbb{E}[Y^2] =$

$\mathbb{E}[XY] =$

$\mathrm{Var}(X) =$

$\mathrm{Var}(Y) =$

$\mathrm{Cov}(X, Y) =$

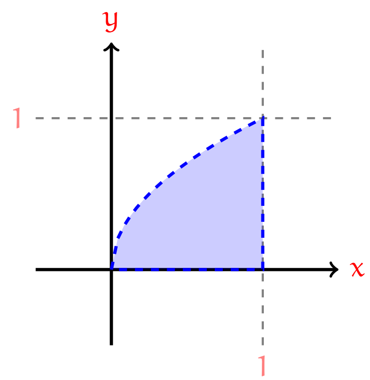
Recall: $\mathrm{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$

$\rho(x, y) = \mathrm{Corr}(X, Y) =$

We have not yet defined the **correlation** Corr, but we will come back once we have.

Example 2. Suppose X and Y are jointly continuous, with joint density function:

$$f(x, y) = \begin{cases} 8x^2y & \text{if } 0 < x < 1 \text{ and } 0 < y < \sqrt{x} \\ 0 & \text{otherwise} \end{cases}$$



Without first finding the marginals, compute:

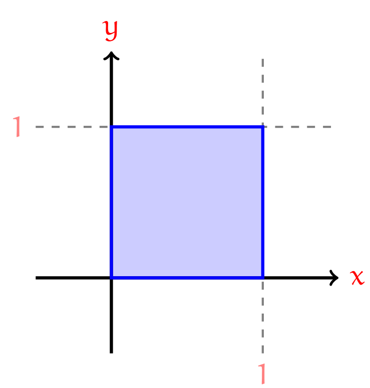
$\mathbb{E}[Y] =$

$\mathbb{E}[XY] =$

Example 3. Let X and Y be independent uniform random variable on $(0, 1)$. Let:

$$Z = \begin{cases} X + Y & \text{if } X \leqslant Y \\ X - Y & \text{if } X > Y \end{cases}$$

Find the expected value of Z.



Recall/derive that the joint density function of two independent uniform random variables on $(0, 1)$ is:

$$f(x, y) = \begin{cases} 1 & \text{if } 0 < x < 1 \text{ and } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

B. **Correlation.** Let's talk about what covariance tells us, by looking at another way to write the covariance of X and Y, using the mean $\mu_X = \mathbb{E}[X]$ of X, and the mean $\mu_Y = \mathbb{E}[Y]$ of X:

$\mathbb{E}\left[(X - \mu_X)(Y - \mu_Y)\right]$

Typically, we normalize, so that the spread from the mean is measured in units of standard deviation. We call this the **correlation**.

Correlation. If $\mathrm{Cov}(X, Y)$ is defined, and if μ_X, μ_Y and σ_X, σ_Y are their means and standard devations, then the **correlation** of X and Y is defined by:

$$\mathrm{Corr}(X, Y) = \rho(X, Y) = \mathbb{E}\left[\frac{X - \mu_X}{\sigma_X} \cdot \frac{Y - \mu_Y}{\sigma_Y}\right] =$$

It measures the strength of **linear** dependence between X and Y, and its values lie in the interval:

$$\leqslant \rho(X, Y) \leqslant$$

and we say that X and Y are **uncorrelated** when $\rho(X, Y) = 0$ given above.

Uncorrelated indicates that there is no linear relation between X and Y, like in the example above. $\rho = 1$ indicates a perfect positive linear dependence, e.g. $Y = mX + b$ with m positive, and $\rho = -1$ indicates a perfect negative linear dependence, e.g. $Y = mX + b$ with m negative.