

Example 1. Let X_1, X_2, X_3 be the lifetimes of 3 lightbulbs, each with a life expectancy of 1 year. You may use that the process in which light bulbs go out is an example of a truncated **birth process**.

Let $X_{(1)}$ be the time at exactly one lightbulb has gone out, $X_{(2)}$ be the time at which exactly two lightbulbs have gone out, and $X_{(3)}$ be the time at which all the lightbulbs have burned out.

Find the correlation between $X_{(2)}$ and $X_{(3)}$.

The birth process notion was introduced in a B3 discussion problem, involving bacteria division. These are generalizations of Poisson processes, with inter-arrival times being independent exponential random variables, but with parameters that can vary after each arrival.

Regarding the times when the bulbs go out, in other words:

$$X_{(1)} \leq X_{(2)} \leq X_{(3)}$$

is the ordering of X_1, X_2, X_3 from least to greatest, which is called the **order statistics** corresponding to the random variables.

Recall that the variance of an exponential random variable X is:

$$\text{Var}(X) = \frac{1}{\lambda^2}$$

and that the covariance of independent random variables is 0.

A. **Indicator Functions and Expectation: Revisited.** We have discussed indicator functions and expectations before, for example applying them to the matching hat problem. Here, we revisit that topic, and apply it to other problems.

Indicator Functions and Expectation. The indicator function $\mathbb{1}_A$ is 1 if the event A occurs, and is 0 otherwise. Further:

$$\mathbb{E}[\mathbb{1}_A] = \mathbb{P}(A)$$

Example 2. 50 people are placed uniformly at random in 200 rooms.

(a) Compute the expected number of rooms containing exactly 2 people.

Recall that a binomial random variable X counts the number of successes k , assuming n independent trials with success rate p , and that:

$$\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

We can think of the number of people who get placed into a given room as “successes”.

(b) What if the probability of each person being placed in the rooms 1–100 is twice the probability of each person being placed in rooms 101–200, and that the placements of people are independent of each other, what is the expected number of rooms containing exactly 2 people?

B. Conditional Expectation: Discrete Conditioning. Let X be a **discrete** random variable, and A be an **event** in that occurs with **positive** probability.

We want to compute the expected value of X , given that the event A has occurred, and call this a **conditional expectation**:

$$\mathbb{E}[X | A] =$$

As a special case, the given event could be the event that **another** discrete random variable Y achieves one of its values, y .

Discrete Conditional Expectation. Let X and Y be **discrete** random variables and y be a value in the support of Y . Then:

$$\mathbb{E}[X | Y = y] =$$

Example 3. Let $X \sim \text{Poisson}(\lambda_1)$ and $Y \sim \text{Poisson}(\lambda_2)$ be independent Poisson random variables. Find:

$$\mathbb{E}[X \mid X + Y = 10] =$$

Recall that the pmf of $X = \text{Poisson}(\lambda)$ is:

$$\mathbb{P}(X = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

In the course of this we find that $X \mid (X + Y = 10)$ is a binomial random variable.