A. **Conditional Expectation: Continuous Conditioning.** Next, we aim to calculuate the expectation of a **continuous** random variable X given that another **continuous** random variable Y achieves one of its values, y.

Let X and Y be **jointly continuous**. We define:

$\mathbb{E}[X \mid Y = y] =$

There is a subtlety to this, the probability that a continuous random Y has an exact value is always 0. We get around this using the conditional density function:

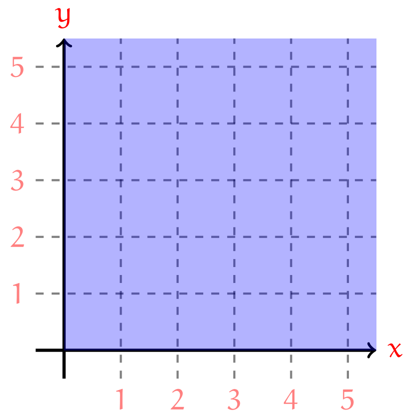
$$f(x \mid y) = \frac{f(x,y)}{f_Y(y)}$$

which is defined as long the probability density of Y at y is nonzero.

Example 1. Let X and Y have joint density function:

$$f(x,y) = \begin{cases} \dfrac{e^{-x/y}e^{-y}}{y} & \text{if } x > 0 \text{ and } y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find: $\mathbb{E}[X \mid Y = y]$.



B. **Tower Property.** Let X and Y be random variables. Then we interpret:

$\mathbb{E}[X \mid Y]$

as a function of the random variable Y via the rule:

$y \mapsto \qquad\qquad\qquad =$

As a function of Y, we can compute its expectation:

$\mathbb{E}\Big[\mathbb{E}[X \mid Y]\Big] =$

This, in fact, applies also in the continuous setting.

Tower Property / Law of Total Expectation. Let X and Y be random variables, where Y is either discrete, or is jointly continuous with X. Then:

$\mathbb{E}\Big[\mathbb{E}[X \mid Y]\Big] =$

In the case Y is **discrete**, this reads as:

$\mathbb{E}[X] =$

In the case X and Y are **jointly continuous**, this reads as:

$\mathbb{E}[X] =$

Intuitively, $\mathbb{E}[X \mid Y = y]$ is our "best" guess for X given that we observe Y to have value y. The tower property in the discrete case says that, if we weight these values by the probability that Y has value y, we obtain $\mathbb{E}[X]$, our "best" guess for X.

Example 2. A miner is trapped in a mine with 3 doors.

Door A leads to safety in 3 hours.

Door B leads the miner back to their original position in 5 hours.

Door C leads the miner back to their original position in 7 hours.

Each door is picked with probability $\frac{1}{3}$ each time.

Let X be the number of hours until the miner walks to safety. Find $\mathbb{E}[X]$.

We call this approach **conditioning** the expecation on the events A, B, and C.