A. X **choose** k**.** In C4 our aim is to generate moments:

kth moment: $\mathbb{E}[X^k]$

For now, we consider the case that X is a **sum of indicator** functions:

$X = \mathbb{1}_{E_1} + \mathbb{1}_{E_2} + \cdots + \mathbb{1}_{E_n} =$

The number of **pairs** of these events that occur is:

Generally:

If X is a **sum of indicator functions** of events:

$X = \mathbb{1}_{E_1} + \mathbb{1}_{E_2} + \cdots + \mathbb{1}_{E_n}$

Then the number of different k–choices of events that occur is:

$\binom{X}{k} =$

Example 1. Let $X \sim \text{Binomial}(n, p)$. Then:

$X =$

Find:

$\mathbb{E}\left[\binom{X}{k}\right] =$

Use the above to re-derive formulas for the first and second moments of $\text{Binomial}(n, p)$.

Example 2. Let's look again at the matching hat problem with n people. Let X be number of people who get their hat back. As usual, we have:

$$X = \mathbb{1}_{E_1} + \mathbb{1}_{E_2} + \cdots + \mathbb{1}_{E_n}$$

where E_i is the event the ith person gets their hat back.

Find:

$$\mathbb{E}\left[\binom{X}{k}\right] =$$

Use the above to find formulas for the first and second moments of X.

B. **Moment Generating Function.** We next describe a function that encapsulates **all** the moments of a random variable.

The **moment generating function** of random variable X is:

$M_X(t) =$

discrete case: $M_X(t) =$

continuous case: $M_X(t) =$

If this expected value is not defined for at least one positive value of t, we say the moment generating function is **undefined**.

If the moment generating function is defined, then it is **analytic** at $t = 0$, meaning it has a Taylor series expansion at 0, and that Taylor series converges to the moment-generating function in some "neighborhood" of 0.

This is how the moment–generating function generates moments:

$M_X(0) =$

$M'_X(0) =$

$M''_X(0) =$

That the derivative can "pass through" $\mathbb{E}$ is dependent on technical aspects about how derivatives interact with infinite sums and integrals. For example, see the Leibniz integral rule.

If the moment generating function of X is defined, then:

$M_X^{(n)}(0) =$

In particular, if the moment generating function is defined, then all moments are defined and finite. This is a strong condition: the Cauchy distribution we had explored earlier (rotating lamp shining on wall) has an undefined expectation, so has no moment-generating function.

Example 3. Let $X \sim \text{Poisson}(\lambda)$. Find the moment generating function of X.

Recall, the pmf of $\text{Poisson}(\lambda)$ is:

$$\mathbb{P}(X = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

where $k \in \mathbb{N}$.

Re-derive the first and second moments of Poisson random variable.

Example 4. Let $X \sim \text{Bernoulli}(p)$. Find the moment generating function of X.

Example 5. Let $X \sim \text{Uniform}(a, b)$. Find the moment generating function of X.

Recall, the pdf of $\text{Uniform}(a, b)$ is:

$$f(x) = \frac{1}{b-a}$$

if x is in (a, b), and is zero otherwise.

Example 6. Let $X \sim \text{Exp}(\lambda)$. Find the moment generating function of X.

Recall, the pdf of $\text{Exp}(\lambda)$ is:

$$f(x) = \lambda e^{-\lambda x}$$

if $x \geqslant 0$, and is zero otherwise.