A. **Poisson Processes.** Consider an arrival process, for example the total number of customers that have arrived to a shop by t minutes after opening. Define:

$N_t =$

A **counting process** $\{N_t\}_{t \geqslant 0}$ satisfies:

initial value: $N_0 =$

nondecreasing: N_t is nondecreasing as t increases

unit jumps: each time N_t changes, it increases by:

In our customer arrival process, suppose that the N_t are **Poisson random variables**, that the customer arrivals are **independent** from each other, and that a expected value of λ customers arrive per minute.

A **Poisson process** with rate parameter λ is a counting process $\{N_t\}_{t \geqslant 0}$ with the following properties.

If $s < t$ then:

independent increments: $N_t - N_s \perp \Big\{ \qquad \Big\}_{r \leqslant s}$

Poisson increments: $N_t - N_s \sim$

and as a consequence, **stationary increments**: $N_t - N_s \sim$

It can be shown that any counting process that satisfies, for all $s < t$, the following properties:

- $N_t - N_s \perp \{N_r\}_{r \leqslant s}$
- $N_t - N_s \sim N_{t-s}$

is a Poisson process for some parameter λ.

Example 1. Let $\{N_t\}_{t\geqslant 0}$ be a Poisson process with positive rate λ. Fix positive times $t > s$ and find:

(a) $\mathbb{P}(N_s = 3, N_t = 5) =$

Recall:

$$\mathbb{P}\left[\text{Poisson}(\lambda) = k\right] = e^{-\lambda}\frac{\lambda^k}{k!}$$

(b) $\mathbb{P}(N_s = k \mid N_t = 5) =$

B. **Exponential Inter–Arrivals.** Consider a Poisson process $\{N_t\}_{t\geqslant 0}$ of arrivals. Let T_1 denote the time until first arrival. Then:

$\mathbb{P}(T_1 > t) =$

In a Poisson process with positive rate λ, the inter–arrival times are:

Therefore, if T_n is the time until the nth arrival, then:

$T_n \sim$

If T_i denotes the time until the ith arrival, then the inter–arrival times are $T_1, T_2 - T_1, T_3 - T_2, \ldots$

C. **Principle of Superposition.** Consider two **independent** Poisson arrival processes, for example arrival process of different kinds of customers to a shop:

$\{X_t\}_{t\geqslant 0}$ with rate parameter $\lambda \longrightarrow$ **returning** customers

$\{Y_t\}_{t\geqslant 0}$ with rate parameter $\mu \longrightarrow$ **new** customers

$\{Z_t\}_{t\geqslant 0} =$ $\longrightarrow$ **all** customers

It can be shown that $\{Z_t\}_{t\geqslant 0}$ is also a counting process with independent arrivals. Let's calculate, for $s < t$, the increment:

The unit jump property is a bit subtle to address in showing that Z_t is a counting process.

$Z_t - Z_s =$

Principle of Superposition. A sum of $\perp$ Poisson processes with rate parameters $\lambda_1, \ldots, \lambda_n$ is a Poisson process with rate parameter:

$\lambda =$