A. **Markov Chains.** Suppose that whether it rains tomorrow **only** depends on the weather today. This divides days into two **states**:

State $0$.

State $1$.

And we call the set of states $\{R, N\}$ the **state space**.

If it rained today, then it rains tomorrow with probability $\frac{3}{5}$ and does not rain tomorrow with probability $\frac{2}{5}$. If it didn't rain today, then it rains tomorrow with probability $\frac{1}{5}$ and does not rain tomorrow with probability $\frac{4}{5}$.

We represent this with a **Markov diagram**:

We use this to assemble a **transition matrix**:

$$P = \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix} = \begin{bmatrix} P_{RR} & P_{RN} \\ P_{NR} & P_{NN} \end{bmatrix} =$$

The transition matrix is an example of a **stochastic matrix**, meaning it has only nonnegative entries, and the sum of entries in each row equals $1$.

Let $X_n$ be the state $n$ days from now. Then:

($\star$) given $X_n$, we have $X_{n+1}$ is (conditionally) independent of:

Because of this property, and because there is a finite and constant state space consisting of disjoint outcomes, we call $\{X_n\}_{n \geqslant 0}$ a **Markov chain**.

Suppose we observed that **today it is raining**, i.e. that $X_0 =$

Then the distribution of probability among initial states is represented by the **row**:

$$\vec{\nu}_0 = \begin{bmatrix} \mathbb{P}(X_0 = R) & \mathbb{P}(X_0 = N) \end{bmatrix} =$$

Here $\nu$ is "nu" and is read as "new".

Let's calculate the distribution of probability among states for tomorrow:

$$\vec{\nu}_1 = \begin{bmatrix} \mathbb{P}(X_1 = R) & \mathbb{P}(X_1 = N) \end{bmatrix} =$$

which we can write using matrix multiplication:

$$\vec{\nu}_1 =$$

Let's calculate the distribution of probability two days from now:

$\vec{v}_2 = \begin{bmatrix} \mathbb{P}(X_2 = R) & \mathbb{P}(X_2 = N) \end{bmatrix}$

which we can also write using matrix multiplication:

$\vec{v}_2 =$

Given an initial distribution of probability among states, $\vec{v}_0$, the distribution of probability $\vec{v}_n$ among states at stage $n$ in the Markov chain is:

For example, let's calculate the probability that it will rain 4 days from now, using a calculator to help us compute:

$$P^4 = \begin{bmatrix} \frac{219}{625} & \frac{406}{625} \\ \frac{203}{625} & \frac{422}{625} \end{bmatrix}$$

So: $\mathbb{P}(X_4 = R) =$

**Example 1.** A Markov chain $\{X_n\}_{n\geqslant 0}$ with states $\{1, 2, 3\}$ has transition matrix:

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

(a) Sketch a Markov diagram.

(b) You uniformly at random select one of the three states to begin from.

Find $\mathbb{E}[X_2]$, i.e. the expected value of stage $2$ in the Markov chain.