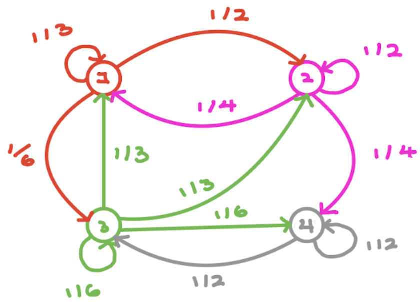
Example 1. A Markov chain $\{X_n\}_{n \geqslant 0}$ with states $\{1, 2, 3, 4\}$ has transition matrix:

$$P = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} & \frac{1}{6} & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$



Assume the initial state is $X_0 = 1$.

(a) What is the probability that State 3 is reached before State 4?

We define:

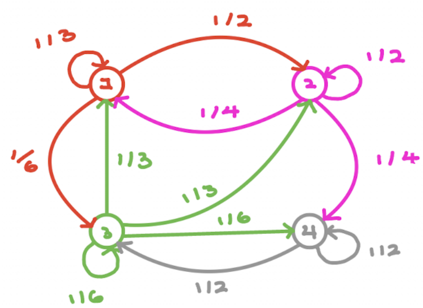
$p_i =$

Immediately we know:

$p_3 =$ and $p_4 =$ and our goal is to find:

We use **first step analysis** to devise a system of equations.

Each equation comes from an application of the law of total probability.



Recall: the initial state was 1.

(b) What is expected number of steps until the chain enters either States 3 or 4?

We define:

$\mu_i =$

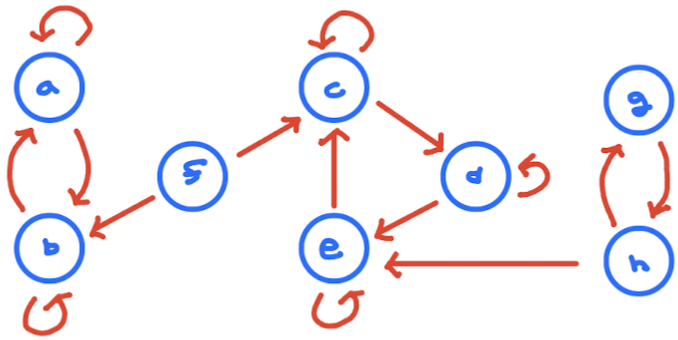
Immediately we know:

$\mu_3 =$ and $\mu_4 =$ and our goal is to find:

We use **first step analysis** to devise a system of equations.

Each equation comes from an application of the law of total expectation, aka the tower property.

A. **Classification of States.** Consider the following Markov diagram, where an arrow indicates a transition that can occur with positive probability.



Because it is possible to eventually transition from state g to state c, we write:

We say that c is **admissible** from g.

On the other hand, in the reverse order:

We say that states a and b **communicate** because:

By definition any state communicates with:

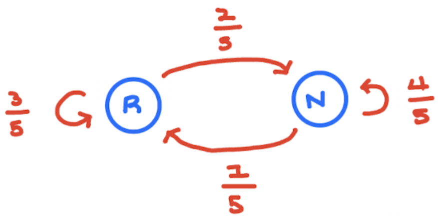
We can thus divide states into **communication classes**:

These are equivalence classes, where the equivalence relation is communication.

A communication class is **transient** if it is possible to transition out of the class and never come back. Otherwise, it is is **recurrent**.

If we allow the chain to run indefinitely, it can be shown that a transient class will only be visited finitely many times.

B. **Ergodicity.** Let's return to the Markov chain $\{X_n\}_{n \geqslant 0}$ of rainy, R, versus not–rainy, N, days.



$$\implies P = \begin{pmatrix} \frac{3}{5} & \frac{2}{5} \\ \frac{1}{5} & \frac{4}{5} \end{pmatrix}$$

Q. Over the long–term, what is the proportion of rainy days versus not–rainy days, in other words, what is what we call the **stationary distribution** $\vec{\pi}$ of probability among states?

$\vec{\pi} =$

In order for $\vec{\pi}$ to be well–defined, it must **not** depend on:

In fact no matter the initial distribution $\vec{v}_0$ we must have:

$\lim_{n \to \infty} \vec{v}_0 P^n =$

As a consequence: $\vec{\pi} P =$

A Markov chain is said to be **ergodic** if the **stationary distribution** $\vec{\pi}$ of the chain is well–defined, in which case:

The stationary distribution is also called the **stationary measure**. The vector $\vec{\pi}$ is a special left–eigenvector for the transition matrix P with eigenvalue $\lambda = 1$.

We will explain later why the chain of rainy versus not–rainy days is ergodic. First, let's assume it is, and then calculate the stationary distribution $\vec{\pi}$.