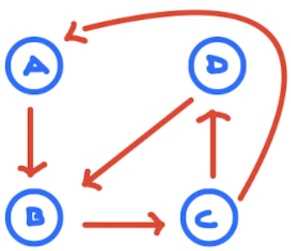
A. **Ergodic Theorem.** Not every Markov chain is ergodic. The key issue preventing a Markov chain from being ergodic is dependence of long–term stages in the chain on the initial state. Consider:

If the initial state was $X_0 = A$ then the probability that $X_n = B$ is:

If the initial state was $X_0 = B$ then the probability that $X_n = B$ is:

So, the associated Markov chain is **not** ergodic. The core issue in this example is that the diagram is **reducible**, meaning:

Consider next:



We can color–code the states using > 1 colors such that the probability of moving from one color to the next color is always 1. Because we needed 3 colors, the Markov diagram is said to be 3–**periodic**.

If if the initial state is X_0 is red, then the probability that X_{3000} is red is:

If if the initial state is X_0 is yellow, then the probability that X_{3000} is red is:

So, the associated Markov chain is **not** ergodic. If such color–coding with > 1 colors is possible, we say the chain is **periodic**.

Ergodic Theorem: Diagram. A Markov chain is ergodic iff it's diagram is:

We will not prove this theorem.

One way to rule out periodicity and reducibility is to find an N so, independent of the initial state, every state can be reached in exactly N steps.

Ergodic Theorem: Matrix. A Markov chain is ergodic iff, there exists N so:

We will also not prove this theorem.

Example 1. Three balls, two red and black, are arranged in a row. At each stage, you select a pair of **adjacent** balls to a swap, where if you have a choice, you are **twice** as likely to swap the red and black than to swap the two reds. Otherwise, you make the swap of adjacent balls uniformly at random.

Show the Markov chain is ergodic, then find its stationary distribution.

Here is a sketch of the states, which we order by the location of the black ball.

The chain is ergodic because there is an N such that we can go from any state to any other state in exactly N steps. Namely:

N =

We can justify this by using matrix multiplication:

Note, that in a check for ergodicity, only the positivity of entries of the transition matrix is relevant.

Next we find the stationary distribution:

Example 2. You repeatedly and indepedently roll a fair 6–sided die. Use Markov chains to do the following. Calculate the probability that you roll two values $\leqslant 4$ in a row before you roll a sequence of one value $\leqslant 4$ followed immediately by one value $\geqslant 5$.

We define:

L :

G :

The outcomes we are asked to consider depend on 2 rolls in a row, so we will use the outcomes of the **last two rolls** as our states:

For an initial state $\star\star$ we define:

$p_{\star\star} =$

Then $p_{LL} =$ and $p_{LG} =$

Next we use first–step analysis.