Intro. Welcome to Math 430! You can call me:

Accessing Content. You can access public course content on our course webpage.

Private content, like your grades, can be accessed through Brightspace.

Be sure to use Brightspace to access our PollEverywhere and Gradescope courses.

Practice problems can be found in MyOpenMath.

Our MyOpenMath course code and passkey are in a Brightspace announcement.

Public Webpage. On the public course webpage you can find the following.

- Announcements: public announcements. Private announcements are in Brightspace.
- Office hours: please stop by whenever you need help!
- Daily Schedule: use it to access handouts and other materials.
- Exam Reference Sheet: provided during exams. Regularly updated.
- Syllabus: read this thoroughly on your own.

Grade Breakdown. There are four grade categories.

- 60% $\rightarrow$ Topic Mastery: Evaluated through midterms and retakes.
- 5% $\rightarrow$ Lecture: Roughly 2 multiple choice polls per lecture.
- 10% $\rightarrow$ Discussion: Groupwork completed in discussion section each week.
- 25% $\rightarrow$ Final Exam

Attendance. In–person attendance of lecture and discussion is required to earn the corresponding credit. A total of 10 dropped polls and 2 dropped groupworks are intended to account for absences due to sickness, family emergency, etcetera.

Recordings. Zoom recordings of lecture will be posted for review, but remote attendance of live lectures is not possible.

Generative AI. Its presence affects the nature of education.

- AI Use: At–home use is encouraged. In–classroom use is prohibited.
- No Homework: AI invalidates homework in its usual form, so there is none.

Practice. Suggested Problems from MyOpenMath are a substitute for homework.

- Posted after each lecture. Work through them regularly.
- Solutions revealed before corresponding exam.
- Feel free to use back–and–forth discussion with generative AI to assist. E.g. for
    - providing hints when you are stuck.
    - explorative questioning, e.g. “If this part were changed, then what?”
    - stripping problems to their cores.
    - revealing connections between the problems and other course content.
    - providing similar problems.
- Exam problems are largely based on suggested problems and lecture examples.

Groupwork problems are drawn from suggested problems.

A. **Well–Ordering Principle.** In this course we study properties of the **integers**.

The set of integers is denoted by the symbol $\mathbb{Z}$.

$\mathbb{Z}$ stands for 'Zahlen' which is the German word for 'numbers'.

$\mathbb{Z} =$

Note the use of **set notation**, a **set** merely being a collection of **elements**.

'Elements' are just 'objects'.

Among the integers, those that are nonnegative form the **natural numbers**:

These numbers certainly don't seem un–natural.

$\mathbb{N} =$

A **subset** of another set consists of some, none, or all, elements of that set.

For example, the subset of $\mathbb{N}$ consisting of positive multiples of 3.

$\{$

Another notable set is the **empty set**.

$\emptyset =$

It is a subset of:

Much of this course will revolve around **proofs**, by which we use logic to establish the validity of mathematical statements.

The tools of logic are based on certain **axioms**, what you might think of as self–evident truths that do not demand proof.

The 'self–evidence' of these truths are in fact up for debate. The study of 'mathematical logic' explores which axioms are truly necessary, and proposes different logical systems based on which axioms we choose. See, for instance, the Zermelo-Fraenkel axioms. An interesting result from the realm of mathematical logic, Gödel's Incompleteness Theorem, states that no matter what (sufficiently complex) axiomatic system we compose, there will always be statements that cannot be established as either true or false.

One of the axioms that will be essential for our study is the following.

The Well–Ordering Principle.

Every nonempty subset of $\mathbb{N}$ has:

in the sense that given such a nonempty subset S of $\mathbb{N}$,

there exists:

such that, for all:

B. **Induction.** An axiom is only useful insofar as it can be applied. We will use ours to establish a basic tool of proofwriting.

Say the following is true about a subset of S of natural numbers.

- **base case:**
- **inductive step:** Whenever k is in S, then:

The **principle of mathematical induction** asserts that then:

However, we need not take this principle of induction as an axiom itself. Nay, we can prove that it is a consequence of the well–ordering principle.

In doing so we will introduce the method of **proof by contradiction**.

This method works as follows. Say you want to prove a statement of the form: "if P, then Q". Begin by assuming that P is true, but Q is **not** true. Using rules of logic, arrive at an obviously false statment. Conclude that if P is true, tne Q must also be true.

Proof. Let S be a subset of $\mathbb{N}$ satisfying the 'base case' and 'inductive step'.

Assume, towards a contradiction, that:

Then the set of natural numbers not in S is:

By the well–ordering principle:

By the 'base case' we know:

Therefore $a - 1$ is:

Because a is the least natural number not in S, it follows that:

By the inductive step:

This is a contradiction, because:

Hence, our original assumption, that $S \neq \mathbb{N}$ must have been:

In other words:

□

Example 1. Let n be a natural number. Use induction to prove the formula:

$$0 + 1 + \cdots + n = \frac{n(n+1)}{2}$$

Solution. We use proof by induction on n.

Base case. Let $n = 0$.

Then the equality reads:

which is obviously true. ✓

Inductive Step. Let nonnegative integer k be given.

Assume the equality holds for $n = k$, i.e. assume:

We need to prove the equality holds $n = k + 1$, i.e. we need to prove:

To that end we begin from the lefthand side and apply the inductive assumption:

$$0 + 1 + \cdots + k + (k+1) =$$

Conclusion. The equality holds for all natural numbers n.

$\square$

Example 2. The factorial of a natural number n is defined by:

$n! =$

For example: $4! =$

with the special case: $0! =$

so that we have the simple relationship:

$(n+1)! =$

Let n be a natural number. Use induction to prove the inequality:

$2^n(n!)^2 \leqslant (2n)!$

Solution. We use proof by induction on n.

Base case. Let $n = 0$.

Then the equality reads:

which is obviously true. ✓

Inductive Step. Let nonnegative integer k be given.

Assume the inequality holds for $n = k$, i.e. assume:

We need to prove the inequality holds $n = k+1$, i.e. we need to prove:

To that end we begin from the lefthand side and apply the inductive assumption:

$2^{k+1}[(k+1)!]^2 =$

Conclusion. The inequality holds for all natural numbers n.

□